

REPRESENTABILITY IN SECOND-ORDER PROPOSITIONAL POLY-MODAL LOGIC

G. ALDO ANTONELLI AND RICHMOND H. THOMASON

Abstract. A propositional system of modal logic is *second-order* if it contains quantifiers $\forall p$ and $\exists p$, which, in the standard interpretation, are construed as ranging over sets of possible worlds (propositions). Most second-order systems of modal logic are highly intractable; for instance, when augmented with propositional quantifiers, K, B, T, K4 and S4 all become effectively equivalent to full second-order logic. An exception is S5, which, being interpretable in monadic second-order logic, is decidable.

In this paper we generalize this framework by allowing multiple modalities. While this does not affect the undecidability of K, B, T, K4 and S4, poly-modal second-order S5 is dramatically more expressive than its mono-modal counterpart. As an example, we establish the definability of the transitive closure of finitely many modal operators. We also take up the decidability issue, and, using a novel encoding of sets of unordered pairs by partitions of the leaves of certain graphs, we show that the second-order propositional logic of two S5 modalities is also equivalent to full second-order logic.

§1. Introduction. It is well known that one can extend the language of classical propositional modal logic by adding second-order quantifiers $\forall p$ and $\exists p$. In the standard interpretation these quantifiers range over all *propositions*, i.e., all sets of possible worlds, whereas in the general interpretation the propositional quantifiers range over some collection of propositions that is closed under operations definable in the language. Many systems of propositional modal logic, when augmented with propositional quantifiers, become highly intractable with respect to the standard semantics. For instance, Fine [1970] shows that the systems K, B, T, K4 and S4 become recursively isomorphic to full second order logic upon adjunction of propositional quantifiers.

The system S5, however, is an exception. Independently, Fine [1970] and Kaplan [1970] show that the second-order version of S5 is decidable. The result is established in Fine [1970] by showing how to “eliminate” propositional quantifiers, a proof strategy already employed by Ackermann [1954, pp. 37–47] to show that each sentence of pure monadic second-order logic is equivalent to a first-order sentence in the pure language of identity. (Since the latter theory has the finite model property,

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monadic second-order logic is decidable.) Kaplan [1970] provides a direct embedding of second-order propositional S5 into monadic second-order logic.

This raises the question of whether the decidability of second-order S5 is due to the symmetric nature of the modality or whether it is due to more peculiar features of S5. Certainly, Kaplan's 1970 reduction of second-order S5 to monadic second-order logic critically exploits the fact that S5 coincides with the set of formulas valid in all frames with a universal modality, i.e., an accessibility relation that holds everywhere. (Such an accessibility relation can then be dispensed with.) Obviously, this fact no longer obtains when two or more S5 modalities are present and can interact.

In this paper, we show that the complexity of poly-modal second-order S5 falls in line with the norm. In particular, poly-modal S5 with propositional quantifiers is dramatically more expressive than its mono-modal counterpart. In proving this result we show how to represent the transitive closure of finitely many modal operators using propositional quantifiers. This device is interesting in its own right, and also figures in our proof of the equivalence of second-order S5 with only two modalities to full second-order logic.

One more point is worth mentioning. The restrictions on S5 modalities preclude the use of the standard encodings of second-order logic that are used to show the undecidability of K, T, B, K4 and S4. Therefore, our reduction has to appeal to a novel encoding of symmetric irreflexive relations (i.e., sets of unordered pairs). We develop an encoding in terms of partitions over certain nodes of a certain *3-tiered graph*, an encoding that might have independent interest.

§2. The language and its semantics. The language \mathcal{L} comprises a denumerable set $\text{Prop} = \{p, q, r, \dots\}$ of propositional variables, connectives \wedge and \neg , the propositional quantifier $\exists p$, and modalities $[i]$, ($i \in I$), for a given index set I . Formulas are obtained from propositional variables by means of connectives, modal operators, or quantifiers. We take the connectives \vee , \rightarrow and the quantifier $\forall p$ to be defined as usual, as well as the dual modalities $\langle i \rangle$ (for $i \in I$).

A *frame* for \mathcal{L} is a tuple $\mathfrak{F} = (W; R_i : i \in I)$, where W is a non-empty set of worlds and the R_i 's are accessibility relations corresponding to the modalities $[i]$. Unless otherwise noted we assume the R_i 's to be equivalence relations.

A *model* for \mathcal{L} is a tuple $\mathfrak{M} = (W; R_i : i \in I; V)$, where $(W; R_i : i \in I)$ is a frame, and $V : \text{Prop} \rightarrow \mathcal{P}(W)$ is a function assigning a *proposition* (i.e., a set of worlds) to each propositional variable. Where the simplification

will cause no confusion, we will use notation that identifies a propositional variable p with the proposition $V(p)$ assigned to it.

DEFINITION 1. Where $X \subseteq W$, $V[X/p](q) = V(q)$ for $q \neq p$ and $V[X/p](p) = X$. Given a model $\mathfrak{M} = (W; R_i : i \in I; V)$, $\mathfrak{M}[X/p] = (W; R_i : i \in I; V[X/p])$.

DEFINITION 2. Define truth of a formula ϕ at a world w in a model \mathfrak{M} , by induction on ϕ :

$$\begin{aligned} \mathfrak{M}, w \models p &\text{ iff } p \in V(w) \\ \mathfrak{M}, w \models \phi \wedge \psi &\text{ iff } \mathfrak{M}, w \models \phi \text{ and } \mathfrak{M}, w \models \psi \\ \mathfrak{M}, w \models \neg\phi &\text{ iff } \mathfrak{M}, w \not\models \phi \\ \mathfrak{M}, w \models [i]\phi &\text{ iff for all } v \in W, \text{ if } R_i(w, v), \text{ then } \mathfrak{M}, v \models \phi \\ \mathfrak{M}, w \models \exists p\phi &\text{ iff for some } X \subseteq W, \mathfrak{M}[X/p], w \models \phi \end{aligned}$$

As usual, a formula ϕ is *valid in a model* \mathfrak{M} , written $\mathfrak{M} \models \phi$, if it is true at every world in \mathfrak{M} ; and it is *valid*, written $\models \phi$, if it is valid in every model. Given a model \mathfrak{M} and a set X of worlds, it is also convenient to abbreviate $\{w' \in X : R_i(w, w')\}$ as $X \upharpoonright_{R_i} w$. When no confusion arises, we will also write $X \upharpoonright_i w$ for $X \upharpoonright_{R_i} w$.

We define:

$$2S5 = \{\phi : \models \phi\}.$$

One of the questions we take up in this paper is the complexity of 2S5.

§3. Transitive closure and connected models. In this section we show that, for certain purposes, we can restrict ourselves to models of a special kind.

DEFINITION 3. Given a model $\mathfrak{M} = (W; R_i : i \in I; V)$, we let

$$\begin{aligned} R^*(x, y) &\Leftrightarrow \\ &\exists i_1 \dots i_n \exists z_1 \dots z_{n-1} [R_{i_1}(x, z_1) \& R_{i_2}(z_1, z_2) \& \dots \& R_{i_n}(z_{n-1}, y)]. \end{aligned}$$

In other words,

$$R^* = \sum_{n \geq 0} \prod_{j=1}^n R_{i_j}$$

Note that if all the R_i relations are equivalence relations, so is R^* .

DEFINITION 4. A model \mathfrak{M} is *connected* if $R^*(w, w')$, for any $w, w' \in W$. A connected model \mathfrak{M} has depth $k > 0$ if and only if any two worlds are no more than k steps away from each other; i.e., if and only if for any

$w, w' \in W$ there are $i_1, \dots, i_{k-1} \in I$ and worlds $v_1, \dots, v_{k-1} \in W$ such that

$$R_1(w, v_1) \ \& \ R_2(v_1, v_2) \ \& \ \dots \ \& \ R_{k-1}(v_{k-1}, w').$$

A connected model has bounded depth if it has depth k for some k .

LEMMA 5. Let \mathfrak{M} be a model, w a world, X a set of worlds, and

$$\begin{aligned} Y &= X \upharpoonright_* w && \text{i.e.,} \\ &= \{w' \in X : R^*(w, w')\}. \end{aligned}$$

Then $\mathfrak{M}[X/p], w \models \phi$ if and only if $\mathfrak{M}[Y/p], w \models \phi$.

PROOF Show by induction on ϕ that for all $w' \in Y$ and all \mathfrak{M} , $\mathfrak{M}[X/p], w' \models \phi$ if and only if $\mathfrak{M}[Y/p], w' \models \phi$. ⊢

THEOREM 6. $2S5 = \{\phi : \phi \text{ is valid in all connected models}\}$.

PROOF. Since connected models are models, if ϕ fails in a connected model it fails in a model, so the inclusion

$$2S5 \subseteq \{\phi : \phi \text{ is valid in all connected models}\}$$

is trivial. For the converse, suppose ϕ fails at a world w in a model $\mathfrak{M} = (W; R_i : i \in I; V)$. Let $W' = \{w' \in W : R^*(w, w')\}$; also, let $R'_i = R_i \upharpoonright W'$ and $V'(p) = V(p) \cap W'$. Finally, let

$$\mathfrak{M}' = (W'; R'_i : i \in I; V').$$

\mathfrak{M}' is connected. By induction on the subformulas ψ of ϕ , one shows that for all $X_1, \dots, X_m \subseteq W'$, all n , all p_1, \dots, p_n , and all $w' \in W$, $\mathfrak{M}[X_1/p_1, \dots, X_n/p_n], w' \models \psi$ if and only if $\mathfrak{M}'[X_1/p_1, \dots, X_n/p_n], w' \models \psi$. Here is the argument for the quantifier case:

$$\begin{aligned} \mathfrak{M}[X_1/p_1, \dots, X_n/p_n], w' \models \exists p \phi & \Rightarrow \\ \exists X \subseteq W \ \mathfrak{M}[X_1/p_1, \dots, X_n/p_n, X/p], w' \models \phi & \Rightarrow \text{By lemma (5)} \\ \exists X \subseteq W \ \mathfrak{M}[X_1/p_1, \dots, X_n/p_n, X \upharpoonright_* w'/p], w' \models \phi & \Rightarrow \text{Ind. hyp.} \\ \exists Y \subseteq W' \ \mathfrak{M}'[X_1/p_1, \dots, X_n/p_n, Y/p], w' \models \phi & \Rightarrow \\ \mathfrak{M}'[X_1/p_1, \dots, X_n/p_n], w' \models \exists p \phi. & \end{aligned}$$

Conversely,

$$\begin{aligned} \mathfrak{M}'[X_1/p_1, \dots, X_n/p_n], w' \models \exists p \phi & \Rightarrow \\ \exists X \subseteq W' \ \mathfrak{M}'[X_1/p_1, \dots, X_n/p_n, X/p], w' \models \phi & \Rightarrow \text{Since } W' \subseteq W \\ \exists X \subseteq W \ \mathfrak{M}'[X_1/p_1, \dots, X_n/p_n, X/p], w' \models \phi & \Rightarrow \text{Ind. hyp.} \\ \exists X \subseteq W \ \mathfrak{M}[X_1/p_1, \dots, X_n/p_n, X/p], w' \models \phi & \Rightarrow \\ \mathfrak{M}[X_1/p_1, \dots, X_n/p_n], w' \models \exists p \phi & \end{aligned}$$

⊢

Although there is no sentence in the language of $2S5$ expressing that a model has bounded depth (not even over connected models), for any

specific k we can express the fact that the model, if connected, has depth k (as long as there are finitely many modalities).

THEOREM 7. Let \mathfrak{M} be a connected model, and $k > 0$. Then there is a sentence $\text{Depth}(k)$ which is valid in \mathfrak{M} if and only if \mathfrak{M} has depth k .

PROOF. Let $\text{Depth}(k)$ be the sentence:

$$\forall p \left[\bigvee_{n \leq k+1} \bigvee_{i_1 \dots i_n \in I} [i_1] \dots [i_n] p \rightarrow \bigvee_{n \leq k} \bigvee_{i_1 \dots i_n \in I} [i_1] \dots [i_n] p \right].$$

Then the sentence $\text{Depth}(k)$ is true at a world w if and only if every world $k + 1$ steps away from w is already k steps away. If the model is indeed connected, this expresses that it has depth k . \dashv

§4. Adding a universal modality. Given a model \mathfrak{M} , an equivalence relation R^u over W is called *quasi-universal* if $R^* \subseteq R^u$; and it is called *universal* if $R^u(x, y)$ holds for every $x, y \in W$. In this section we show how to axiomatize a quasi-universal modality (universal over connected models).

Suppose the language \mathcal{L} comprises modalities $[i]$ for $i \in I$ and an additional modality $[u]$. Let Un be the set comprising the axioms:

$$\begin{aligned} & \forall p ([u]p \rightarrow [i]p); & \text{for each } i \in I \\ & \forall p ([u]p \rightarrow p) \\ & \forall p ([u]p \rightarrow [u][u]p) \\ & \forall p (p \rightarrow [u]\langle u \rangle p) \end{aligned}$$

LEMMA 8. Let \mathfrak{M} be a model, and $[i]$ any modality in the language of \mathfrak{M} . Then:

- (1) If $\mathfrak{M}, w \models \forall p ([i]p \rightarrow p)$ for all $w \in W$ then R_i is reflexive;
- (2) if $\mathfrak{M}, w \models \forall p ([i]p \rightarrow [i][i]p)$ for all $w \in W$ for all $w \in W$ then R_i is transitive;
- (3) if $\mathfrak{M}, w \models \forall p (p \rightarrow [i]\langle i \rangle p)$ for all $w \in W$ then R_i is symmetric.

PROOF. We consider only case (2). The sentence $\forall p ([i]p \rightarrow [i][i]p)$ holds at a world w if and only if for every proposition p (i.e., for every $p \subseteq W$), $\{w' : R_i(w, w')\} \subseteq p$ implies

$$\{w' : \exists v \in W (R_i(w, v) \& R_i(v, w'))\} \subseteq p.$$

Now let x, y, z be worlds such that $R_i(x, y)$ and $R_i(y, z)$, and let $p = \{w' : R_i(x, w')\}$. Then by hypothesis $z \in p$ so that $R_i(x, z)$, as desired. \dashv

The previous lemma highlights the difference between the “correspondence theory” for standard propositional modal logic and the second-order case. Correspondence theory studies the connections between the modal axioms and the properties of the accessibility relation among possible worlds. For instance the **K5** propositional axiom

$$\diamond p \rightarrow \Box \diamond p$$

is valid in a frame if and only if the accessibility relation R of the frame is *Euclidean*, i.e., satisfying

$$\forall x \forall y \forall z (Rxy \ \& \ Rxz \rightarrow Ryz).$$

(Recall that an axiom is valid in a frame if it is valid in every model based on the frame.) But notice that this and similar correspondence results only hold at the level of *frames*: it is not true that, e.g., the **K5** axiom is true in a *model* if and only if the accessibility relation of the model is Euclidean (there are non-Euclidean models where the axiom holds).

However, the second-order propositional version of the axioms characterizes properties of the accessibility relation at the level of models. As in Lemma (8), the sentence

$$\forall p (\diamond p \rightarrow \Box \diamond p)$$

is valid in a model if and only if the accessibility relation of the model is Euclidean.

THEOREM 9. Let \mathfrak{M} be a model for \mathcal{L} such that $\mathfrak{M} \models \text{Un}$. Then R^u is a quasi-universal **S5** modality in \mathfrak{M} . Moreover, if the model is connected, then R^u is in fact universal.

PROOF. Let u, v be arbitrary worlds in W such that $R^*(u, v)$. Then there are $i_1, \dots, i_n \in I$ and worlds w_1, \dots, w_{n-1} such that

$$R_{i_1}(u, w_1) \ \& \ \dots \ \& \ R_{i_n}(w_{n-1}, v).$$

Since $\forall p ([u]p \rightarrow [i]p)$ is valid in \mathfrak{M} , for each $j \in \{1 \dots n\}$ we have $R_{i_j} \subseteq R_u$. Hence, $R_u^n(u, v)$. Since R_u is transitive, $R_u(u, v)$, as desired. The second claim follows immediately from the fact that in a connected model we have $R^*(u, v)$ for any worlds u, v . \dashv

§5. Axiomatizing R^* for finitely many modalities. In this section we show how to represent the “transitive closure” of the modalities $[i]$, under the assumption that there are only finitely many of them. This is relevant to interpretations of $[i]$ as an epistemic operator, for then the modality corresponding to $[i]$ is naturally interpreted as common knowledge (see Fagin, Halpern, Moses, and Vardi [1995]). We begin by establishing that R^* is explicitly definable over bounded-depth connected models.

THEOREM 10. Let \mathfrak{M} be a bounded-depth connected model for a language comprising finitely many modalities $[i]$ for $i \in I$. Then the modality R^* is explicitly definable over \mathfrak{M} .

PROOF. Let \mathfrak{M} have depth k , and let $[*]A$ abbreviate:

$$\bigvee_{n \leq k} \bigvee_{i_1, \dots, i_n \in I} [i_1] \dots [i_n]A$$

Then, $[*]A$ holds at a world w if and only if A holds at a world a finite distance from w . \dashv

The more difficult question is whether R^* is definable over arbitrary models. The following result apparently presents an obstacle to this task.

THEOREM 11. Let \mathfrak{M} be a model, w be a world, and for every n let

$$W_w^n = \{w' \in W : \exists m \leq n \exists i_1 \dots i_m \in I \exists v_1 \dots v_{m-1} \in W \\ R_{i_1}(w, v_1) \& R_{i_2}(v_1, v_2) \& \dots \& R_{i_m}(v_{m-1}, w')\}$$

(so W_w^n is the set of worlds no more than n steps away from w). Let \mathfrak{M}_w^n be the model whose worlds are W_w^n , whose accessibility relations are the restrictions to W_w^n of the corresponding relations in \mathfrak{M} , and whose valuation V_w^n is such that $V_w^n(p) = V(p) \cap W_w^n$. Then for every formula ϕ there is a number k such that for every world w ,

$$\mathfrak{M}, w \models \phi \iff \mathfrak{M}_w^k, w \models \phi.$$

PROOF by induction on ϕ . \dashv

The import of the theorem is that no formula ϕ can depend on worlds arbitrarily far away from a base world w . This fact is relevant for the epistemic interpretation of the modal operators and the definability of common knowledge between a number of agents. Under the epistemic interpretation of the operators, a proposition p is said to be *common knowledge* between agents 1 and 2 if and only if the following infinitary sentence is true:

$$p \wedge [1]p \wedge [2]p \wedge [1][2]p \wedge [2][1]p \wedge [1][2][1]p \wedge \dots,$$

where the modal prefixes range over all possible finite alternations of $[1]$ and $[2]$. (If we abbreviate the above sentence as $[*]p$, then we see that common knowledge corresponds exactly to the R^* relation.) Since the modal prefixes are arbitrarily long, the truth of the above sentence at a world w depends on worlds arbitrarily distant from w . The theorem above says that common knowledge for two agents cannot be defined for two agents, even using propositional quantifiers, in a language containing only two primitive modalities, representing the knowledge of each of the two agents. However, if an additional primitive modality is available, we can

use it to *axiomatize* a quasi-universal modality, and common knowledge for a finite collection of agents becomes definable.

Our proof of the equivalence of 2S5 to second-order logic assumes a quasi-universal modality. If enough modalities are available, the axiomatization techniques presented in this section yield such a modality. However, since our proof involves special models in which every world is accessible in at most three steps from a base world, we can actually define the R^* modality when only two primitive modalities are available. This is the technique we use in the proof presented in §7.

We are going to axiomatize a modality $[*]$ and its dual $\langle * \rangle$. Intuitively, the formula $[*]p$ holds at a world w if and only if p holds at every world a finite distance from w ; and $\langle * \rangle p$ holds at a world w if and only if p holds at some world a finite distance from w .

It should be noted that in the case of a connected model, $[*]$ is just the universal modality. The situation is different in the general case: for then R_u will only be a subset of R^* . In view of the results of §4, we can assume a quasi-universal modality $[u]$ without any loss in generality.

DEFINITION 12. Suppose that $[u]$ is a quasi-universal modality; then $\text{Sngl}(p)$ abbreviates

$$\langle u \rangle p \wedge \forall q [[u](p \rightarrow q) \vee [u](p \rightarrow \neg q)].$$

LEMMA 13. Let \mathfrak{M} be a model, R_u a quasi-universal modality, and w be a world. Then $\mathfrak{M}, w \models \text{Sngl}(p)$ if and only if p contains exactly one world R_u -accessible from w .

PROOF. First we observe that $\langle u \rangle p$ holds at w if and only if p contains a world R_u -accessible from w .

Moreover for any proposition q , $[u](p \rightarrow q)$ holds at w if and only if $p \upharpoonright_u w \subseteq q$, and similarly for $[u](p \rightarrow \neg q)$.

Suppose for contradiction that p contains two distinct worlds w' and w'' that are R_u -accessible from w , and let $q = \{w'\}$. Then both $p \upharpoonright_u w \subseteq q$ and $p \upharpoonright_u w \subseteq W \setminus q$ fail, contrary to the hypothesis. \neg

DEFINITION 14. Let $[u]$ be quasi-universal; then $\text{Cl}(q, i)$ abbreviates

$$[u](q \rightarrow [i]q).$$

Similarly, $\text{Cl}(q)$ abbreviates

$$\bigwedge_{i \in I} \text{Cl}(q, i).$$

LEMMA 15. Let \mathfrak{M} be a model, R_u be a quasi-universal modality, w be a world and $Y = q \upharpoonright_* w$. Then $\text{Cl}(q, i)$ holds at w if and only if Y is closed under R_i .

PROOF. First, suppose that $\mathfrak{M}, w \models \text{Cl}(q, i)$. We need to show that if $u \in Y$ and $R_i(u, v)$, then $v \in Y$. If $u \in Y$, then $R_*(w, u)$; since $[u]$ is quasi-universal, also $R_u(w, u)$, and hence $\mathfrak{M}, u \models (q \rightarrow [i]q)$.

From $u \in Y$ we have $u \in q$; hence, $\mathfrak{M}, u \models [i]q$. Since $R_i(u, v)$, also $v \in q$, as desired.

Second, suppose that $\mathfrak{M}, w \models \langle u \rangle [q \wedge \langle i \rangle \neg q]$. Then for some u, v , $R_u(w, u)$, $u \in q$, $R_i(u, v)$ and $v \notin q$. Therefore $u \in Y$ and $R_i(u, v)$, but $v \notin Y$. \dashv

We are now ready to define the modality $[*]$ explicitly.

DEFINITION 16. Where A is any formula, $[*]A$ stands for

$$\forall p [\text{Sngl}(p) \wedge \forall q [q \wedge \text{Cl}(q)] \rightarrow \langle u \rangle (p \wedge q)] \rightarrow \langle u \rangle (p \wedge A)];$$

Similarly, $\langle * \rangle A$ stands for $\neg [*] \neg A$.

THEOREM 17. $\mathfrak{M}, w \models [*]A$ if and only if A holds at every world that is R^* -accessible from w .

PROOF. First, assume $[*]A$ holds at w , and let w' be any world that is R^* -accessible from w . We need to show that A holds at w' . Now consider $p = \{w'\}$: then $\text{Sngl}(p)$ holds at w , and moreover p intersects every set of worlds containing w and closed under all the R_i 's (since w' belongs to every set of worlds containing w and closed under all the R_i 's). So the antecedent of $[*]A$ holds at w . We conclude that p intersects the proposition expressed by A , i.e., A holds at w' .

Conversely, suppose that for some p , $\text{Sngl}(p)$, $\forall q [q \wedge \text{Cl}(q)] \rightarrow \langle i \rangle (p \wedge q)$, and $[u](p \rightarrow \neg A)$ all hold at w . Then for some w' , $p = \{w'\}$ and, where $X = \{w'' : R_*(w, w'')\}$, we have $w \in X$ and X is closed under R_* . Therefore, $w' \in X$, i.e., $R_*(w, w')$. Furthermore, $R_u(w, w')$, and therefore $\neg A$ holds at w . \dashv

§6. Coding for pairs. In this section we take up the task of showing that 2S5 is not recursively axiomatizable. In fact, as we will see, it is equivalent to full second-order logic. In view of the result of §3, we can immediately reduce this problem to that of showing that the set of sentences valid in all *connected* models is equivalent to full second-order logic.

Further, the set of second-order validities itself can be reduced to a smaller set of validities, in virtue of the following theorem.

THEOREM 18 (Nerode and Shore [1980]). Full second-order logic is recursively equivalent to second-order logic with quantification restricted to symmetric and irreflexive binary relations, over domains with at least two elements.

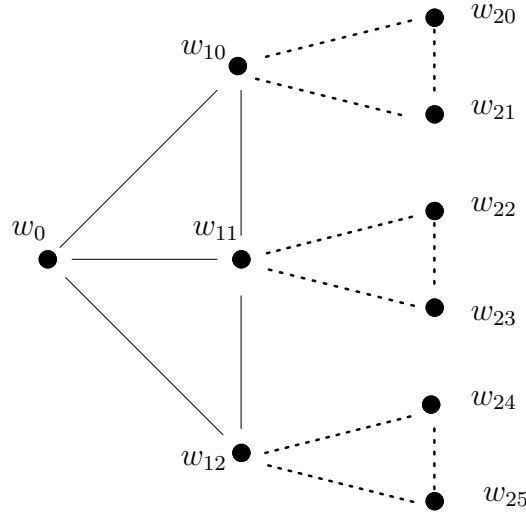


FIGURE 1. Coding for pairs, first try.

Symmetric and irreflexive binary relations are just sets of unordered pairs (i.e., doubletons; from now and unless otherwise noted simply ‘pairs’). The result of Nerode and Shore states that second-order logic (with the standard interpretation) is reducible to second-order logic in which the relational variables range over sets of doubletons. Following Kremer [1997], let us refer to this logic as 2-SIB. We will show how to effectively reduce the set of 2S5-sentences valid in all connected models to 2-SIB. In particular, we will show how to turn any sentence ϕ of 2-SIB into a sentence ϕ' of 2S5 in such a way that ϕ is valid if and only if ϕ' is. Kremer [1997] offers a corresponding reduction for intuitionistic logic with propositional quantifiers.

The main idea is to use quantification over propositions (sets of worlds) to represent quantification over symmetric irreflexive relations. This requires that we assign to each pair of worlds a “representative” or “code,” in such a way that quantification over pairs can be simulated by quantifying over their codes.

A first attempt is depicted in Figure 1, where the solid lines represent, say the [1]-modality and the dotted lines represent the [2]-modality. Worlds in the frame of Figure 1 can be subdivided into 3 tiers. Tier 1 comprises the single world w_0 ; tier 2 comprises the 3 worlds w_{10}, w_{11}, w_{12} ; and tier 3 comprises the 6 worlds w_{20}, \dots, w_{25} . The idea is that the worlds in tier 2 act as “codes” for the pairs of worlds in tier 3. So, for instance, w_{11} codes the pair $\{w_{22}, w_{23}\}$, and the proposition $\{w_{10}, w_{12}\}$ codes the symmetric irreflexive relation $\{\{w_{20}, w_{21}\}, \{w_{24}, w_{25}\}\}$.

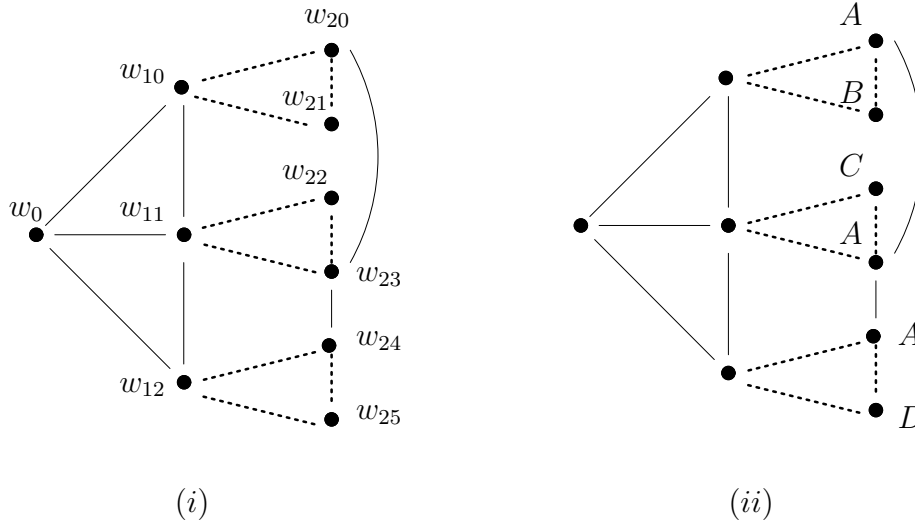


FIGURE 2. Coding for pairs: with “jumpers,” and re-labelled.

We can see then that by quantifying over sets of tier-2 worlds we can represent quantification over sets of pairs of tier-3 worlds. The single tier-1 world allows us to distinguish pairs from their codes. First notice that w_0 is characterized by not being R_2 -related to anything (other than itself), so that the tier-2 worlds are uniquely identified by their being R_1 -related to w_0 ; in turn, tier-3 worlds are R_2 -related to tier-2 worlds.

There is one obvious problem with this idea, namely the fact that in this way we can only represent sets of non-overlapping (disjoint) pairs. The way around this obstacle is to increase the number of disjoint pairs, but in turn to identify some of the members, by having “jumpers” between worlds (see Figure 2 (i)). Two items are worth noting in this respect: first, that although it might be more natural to introduce a third modality for this purpose, we can get away with using R_1 again, thereby obtaining an optimal result; second, given that we use an S5-modality to identify members of different pairs, the worlds thus identified form a totally connected subset (a “clique”), as is natural if R_1 is to function as *ersatz* for true identity.

Once worlds are identified by jumpers, we can have codes for non-disjoint pairs, as we can see when the model of Figure 2 (i) is re-labelled as in (ii). In this case, for instance, the model contains codes for the pairs $\{A, B\}$, $\{A, C\}$, and $\{A, D\}$.

One more step is necessary in order to fully represent quantification over sets of pairs by quantification over their codes: we need to make sure that every pair has a code. For instance, to take up a finite case, over a domain

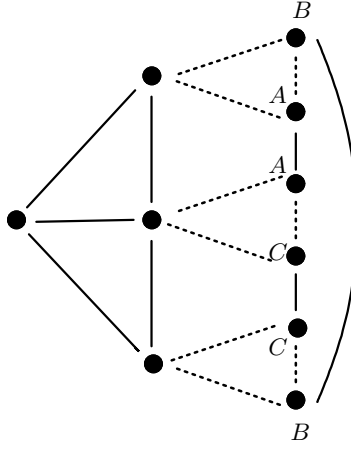


FIGURE 3. Coding for pairs over $\{A, B, C\}$: a full model.

of n objects ($n \geq 2$), there are $\frac{n(n-1)}{2}$ pairs, which can be represented by having $n(n-1)$ “new objects”, partitioned into n equivalence classes, each of cardinality $n-1$.

This leads to the introduction of the notion of a *full* model. Intuitively, this is a model whose “jumpers” are so configured that the resulting codes represent all pairs over some domain. A full model over a 3-membered set is given in Figure 3.

A word on notation, before we proceed with the definitions. For any set X , let us denote the set of all pairs (cardinality-2 subsets) of X by $[X]^2$. Moreover, where $G = (V, E)$ is a graph, for any $x, y \in V$ let $a(x, y)$ be the unique z such that $\{z, x\}, \{z, y\} \in E$, where there is such a unique z ; otherwise, $a(x, y)$ is undefined.

DEFINITION 19. A (nondirected) graph $G = (V, E)$ is *3-tiered* if and only if:

1. $V = \{w_0\} \cup T_1 \cup T_2$, with $\{w_0\}$, T_1 , and T_2 pairwise disjoint.
2. $\{w, w'\} \in E$ for all $w, w' \in \{w_0\} \cup T_1$.
3. For all $w \in T_2$ there are exactly two $w' \in T_3$ such that $\{w, w'\} \in E$.
4. There is an equivalence relation \cong over T_3 such that for all $w, w' \in T_3$, $\{w, w'\} \in E$ if and only if $w \cong w'$ and $w \neq w'$.
5. For all $w, w' \in T_2$, if $w \neq w'$, $\{w, x\} \in E$, and $\{w', x'\} \in E$, then $x \neq x'$.

DEFINITION 20. A model is *3-tiered* if and only if there is a 3-tiered graph $G = (V, E)$ such that:

1. $V=W$.
2. For all $w, w' \in W$, $R_1(w, w')$ if and only if either (i) $w, w' \in \{w_0\} \cup T_2$ or (ii) $w, w' \in T_3$ and $w = w'$ or $\{w, w'\} \in E$.

3. For all $w, w' \in W$, $R_2(w, w')$ if and only if either (i) $w = w'$, or (ii) $\{w, w'\} = \{x, x'\}$, where $x \in T_2$, $x' \in T_3$, and $\{x, x'\} \in E$, or (iii) $w, w' \in T_3$ and $a(w, w') \in T_2$ is defined.

Note that if \mathfrak{M} is a model then R_1 and R_2 must be equivalence relations.

DEFINITION 21. Let $|X| \geq 2$. \mathfrak{M} represents $[X]^2$ if and only if there are functions f from T_2 onto $[X]^2$ and h from T_3 onto X such that

1. $f(a(x, y)) = \{h(x), h(y)\}$
2. $x \cong y$ if and only if $g(x) = g(y)$;

where \cong is the equivalence relation associated with a 3-tiered model \mathfrak{M} .

Clearly, if a model represents $[X]^2$ then it represents $[Y]^2$ for any Y of the same cardinality as X .

DEFINITION 22. A full 3-tiered model meets the following two conditions.

1. If $a(x, y) \in T_2$ and $a \neq y$ then $x \not\cong y$.
2. If $x \not\cong y$ for $x, y \in T_3$, then there is exactly one pair $\{x', y'\}$ such that $x' \cong x$, $y' \cong y$, and $a(x', y') \in T_2$.

THEOREM 23. Let \mathfrak{M} be a full 3-tiered model. Then there is a set X such that \mathfrak{M} represents $[X]^2$.

PROOF. Let X be the collection of all \cong -equivalence classes over T_3 , and let h be the canonical homomorphism from T_3 to X . Where $w \in T_2$, let $f(w)$ be $\{h(x) : \{w, x\} \in E\}$. In view of Def. 19 and (1) of Def. 22, $|f(w)| = 2$ and $f(w) \subseteq X$; so $f(w) \in [X]^2$.

First, we show that f is onto $[X]^2$. Suppose that $\{x, y\} \in [X]^2$. Then $x \not\cong y$, and so by (2) of Def. 22, there are x', y' with $a(x', y') \in T_2$ and $x' \cong x$, $y' \cong y$. And we have $f(a(x', y')) = \{x, y\}$.

Second, we show that f is one-one. Suppose that $f(w) = f(w')$. Then we have $\{w, x_1\} \in E$, $\{w, x'_1\} \in E$, $x_1 \cong x'_1$, and $\{w, x_2\} \in E$, $\{w, x'_2\} \in E$, $x_2 \cong x'_2$, where $x_1 \neq x'_1$ and $x_2 \neq x'_2$. By (1), $x_1 \not\cong x_2$. So by (2), $x_1 = x'_1$ and $x_2 = x'_2$. Then by Def. 19, $w = w'$. \dashv

THEOREM 24. Let X be a set; then there is a full 3-tiered model that represents $[X]^2$.

PROOF. Let $T_2 = [X]^2$, let

$$T_3 = \{ \{ \{x, y\}, x, a \} : \{x, y\} \in [X]^2 \} \cup \{ \{ \{x, y\}, y, a \} : \{x, y\} \in [X]^2 \},$$

where $a \notin X$. (The only purpose of a is to ensure that T_2 and T_3 are disjoint.) Let $w_0 \notin T_2, \notin T_3$. Let $\{w, w'\} \in E$ for all $w, w' \in \{w_0\} \cup T_2$. For $\{x, y\} \in T_2$, let $\{ \{x, y\}, A \} \in E$ if and only if $A = \{ \{x, y\}, x, a \}$ or $A = \{ \{x, y\}, y, a \}$. It is straightforward to check that this defines a 3-tiered graph.

Define R_1 over $\{w_0\} \cup T_2$ and R_2 over $T_2 \cup T_3$ in such a way as to satisfy Def. 20. Further, let $\{\{x, y\}, z, a\}$ and $\{\{x', y'\}, z', a\}$ in T_3 be R_1 -related if and only if $z = z'$. In this way we obtain a 3-tiered model.

Let $h(\{\{x, y\}, z, a\}) = z$, and let

$$\begin{aligned} f(\{\{x, y\}\}) &= \{h(\{\{x, y\}, x, a\}), h(\{\{x, y\}, y, a\})\} \\ &= \{x, y\}. \end{aligned}$$

Again, it is straightforward to check that these choices show \mathfrak{M} to represent $[X]^2$.

Finally, the model satisfies the two conditions of Def. 22 and so is full. \dashv

THEOREM 25. Let \mathfrak{M} be a 3-tiered model. Then \mathfrak{M} is full if and only if there is a set X such that \mathfrak{M} represents $[X]^2$.

PROOF. In view of Theorem 23, we only need to show that if \mathfrak{M} represents $[X]^2$ then \mathfrak{M} is full.

Assume that \mathfrak{M} represents $[X]^2$, and let $a(x, y) \in T_2$. Now $f(a(x, y)) = \{h(x), h(y)\}$. But if $x \cong y$, $f(a(x, y))$ would be a singleton and so could not be a member of $[X]^2$. So $x \not\cong y$. This verifies Condition (1) of Def. 22.

Suppose $x \not\cong y$, where $x, y \in T_3$. Then $h(x) \neq h(y)$, so $\{h(x), h(y)\} \in [X]^2$. So $\{h(x), h(y)\} = f(a(x, y))$, where of course $a(x, y) \in T_2$. Now suppose that $x' \cong x$, $y' \cong y$, and $a(x', y') \in T_2$, and also that $x'' \cong x$, $y'' \cong y$, and $a(x'', y'') \in T_2$. Then $h(x'') = h(x') = h(x)$ and $h(y'') = h(y') = h(y)$. So $f(a(x', y')) = \{h(x), h(y)\} = f(a(x'', y''))$. Since f is one-one, $a(x', y') = a(x'', y'')$; but then $x' = x''$ and $y' = y''$. This verifies Condition (2) of Def. 22. \dashv

We are now in a position to see that any symmetric, irreflexive relation over a set X can be identified with a set p of tier-2 worlds in the corresponding full model. In particular, if p is such a set of tier-2 worlds, and x, y are tier-3 worlds, to say that x and y stand in the relation represented by p is to say that there are sibling tier-2 worlds $x' \cong x$ and $y' \cong y$ whose common parent w is a member of p .

§7. Non-axiomatizability. In this section we undertake the detailed work of formalizing the results of the previous section. We work in a second-order propositional language containing two S5 modalities [1] and [2] (corresponding to R_1 and R_2). This language may contain other primitive modalities, but our proof does not require any such modalities. With no loss of generality, we expand the language by adding two propositional constants: r , which will denote the singleton proposition containing the root of the full model under consideration; and \perp , which will denote the empty proposition. (This is an *inessential extension* in

the sense of Rabin [1965, p. 59]; it is well known that such an extension does not change the complexity of the set of validities—see, for instance, Tarski, Mostowski, and Robinson [1953, p. 16].) The semantic definitions of §2 are extended in the obvious way to allow for constants.

Our axiomatization will require the presence of a modality $[*]$ corresponding to the transitive closure of the sum of R_1 and R_2 in a 3-tiered modal model \mathfrak{M} . Since all such models are connected with depth $k = 3$, by Theorem 10, such a modality is explicitly definable.

7.1. Axiomatizing full models. The first task is to provide a formal characterization of full models. In particular, we will identify a sentence ϕ which is valid in a (connected) model if and only if the model is full. We proceed in stages, by introducing a number of abbreviations and verifying at each stage that they have the intended meaning over connected models where $[*]$ is quasi-universal.

1. Any two worlds are no more than 3 steps away: this is the sentence $\text{Depth}(3)$ of Theorem 7.
2. Identity of propositions: $p = q$ abbreviates $[*](p \leftrightarrow q)$. Clearly $p = q$ is satisfied in a world model if and only if the same worlds belong to both p and q , and hence if and only if p is the same set as q . Similarly, $p \subseteq q$ abbreviates $[*](p \rightarrow q)$.
3. Some world in p is R_i related to some world in q ($i = 1, 2$):

$$\langle * \rangle (p \wedge \langle i \rangle q);$$

this is especially significant when p and q are singletons, effectively allowing us to say that two worlds are related; further, when the relations are symmetric, this is the same as $[*](q \wedge \langle i \rangle p)$. Abusing language somewhat, we abbreviate the last displayed formula as $R_i(p, q)$.

4. Given a formula $\theta(p)$, we abbreviate

$$\exists p \forall q (\theta(q) \leftrightarrow p = q)$$

as $\exists! p \theta(p)$.

5. Proposition r is the tier-1 world of the model we are characterizing: r is R_1 -connected to at least one world (other than itself) and it is not R_2 -connected to any other worlds. Let $\text{Root}(r)$ abbreviate the sentence:

$$\text{Sngl}(r) \wedge \exists p [\text{Sngl}(p) \wedge p \neq r \wedge R_1(r, p)] \wedge \forall q (R_2(r, q) \rightarrow r = q).$$

6. The world r is the unique world with this property:

$$\forall p (\text{Root}(p) \leftrightarrow p = r).$$

7. A world w is a tier-2 world if and only if it is distinct from r but R_1 -connected to it. Given that we cannot quantify directly over worlds,

we formalize this by saying that every world in p is a tier-2 world, which we abbreviate as $T_2(p)$:

$$\forall q(\text{Sngl}(q) \wedge q \subseteq p \rightarrow q \neq r \wedge R_1(r, q)).$$

8. A proposition p contains only tier-3 worlds if and only if every $w \in p$ is R_2 -connected to a tier-2 world. Thus, $T_3(p)$ abbreviates:

$$\forall q(\text{Sngl}(q) \wedge q \subseteq p \rightarrow \exists s(T_2(s) \wedge R_2(q, s))).$$

9. Every world other than r is either tier-2 or tier-3:

$$\forall q(\text{Sngl}(q) \wedge q \neq r \rightarrow T_2(q) \vee T_3(q)).$$

10. No tier-3 world is R_1 -connected to a tier-2 world:

$$\forall q(\text{Sngl}(q) \wedge T_3(q) \rightarrow \forall p(T_2(p) \rightarrow \neg R_2(q, p))).$$

11. Every tier-2 world is R_2 -connected to exactly two tier-3 worlds:

$$\begin{aligned} \forall p[T_2(p) \wedge \text{Sngl}(p) \rightarrow \\ \exists q_1, q_2(q_1 \neq p \wedge q_2 \neq p \wedge q_1 \neq q_2 \wedge R_2(p, q_1) \wedge R_2(p, q_2) \wedge \\ \forall r(R_2(p, r) \rightarrow r = q_1 \vee r = q_2))]. \end{aligned}$$

Now let $\mathbf{3-Tier}(r)$ specify that the model is a 3-tiered graph rooted at r , by saying that the model has depth 3, that r is the unique proposition such that $\text{Root}(r)$, that every tier-2 world is R_2 -connected to exactly two tier-3 worlds, and that every world other than r is either tier-2 or tier-3.

We now deal with the fullness condition.

1. $\text{sib}_2(p, q)$ expresses that p and q have a common R_2 -parent in tier-2:

$$\exists r(T_2(r) \wedge R_2(r, p) \wedge R_2(r, q)).$$

2. no two R_2 -siblings in tier-3 are R_1 -related:

$$\forall p, q(T_3(p) \wedge T_3(q) \wedge \text{sib}_2(p, q) \rightarrow \neg R_1(p, q)).$$

3. p and q are tier-3 worlds such that $p \cong q$ ($\text{sim}(p, q)$):

$$\text{Sngl}(p) \wedge p \wedge \text{Sngl}(q) \wedge T_3(p) \wedge T_3(q) \wedge R_1(p, q).$$

4. no two tier-3 siblings are R_1 -related:

$$\forall p, q[[T_3(p) \wedge T_3(q) \wedge \text{Sngl}(p) \wedge \text{Sngl}(q) \wedge \text{sib}_2(p, q)] \rightarrow \neg R_1(p, q)].$$

5. for any two distinct R_1 -equivalence classes A and B over tier-3 there is exactly one pair of worlds $x \in A$ and $y \in B$ that are R_2 -siblings:

$$\begin{aligned} \forall p \forall q[[\text{Sngl}(p) \wedge \text{Sngl}(q) \wedge \neg \text{sim}(p, q)] \rightarrow \\ \exists p' \exists q' \forall p'' \forall q''[[\text{sim}(p, p') \wedge \text{sim}(q, q') \wedge \text{sib}_2(q, q') \leftrightarrow \\ [p' = p'' \wedge q' = q'']]]. \end{aligned}$$

Finally, let FullModel be the sentence conjoining $\mathbf{3-Tier}(r)$ with the fullness conditions that (i) if $a(x, y) \in T_2$ then $x \not\cong y$ and (ii) If $x \not\cong y$ for $x, y \in T_3$, then there is exactly one pair $\{x, y\}$ such that $x' \cong x$, $y' \cong y$, and $a(x', y') \in T_2$. Thus, we have established the following theorem.

THEOREM 26. Let \mathfrak{M} be a connected model; then **FullModel** is valid in \mathfrak{M} if and only if \mathfrak{M} is a full model.

7.2. Translating 2-SIB. Recall that 2-SIB is second-order logic in which the relational variables range over the collection of all symmetric, irreflexive relations over the domain (with the cardinality of the domain at least 2). In particular, the formulas of 2-SIB are obtained from binary relation symbols by means of the connectives \neg and \wedge and the quantifiers $\exists x$ and $\exists R$. We need not add identity to the list of the basic predicates because in the standard interpretation of 2-SIB, $x = y$ is definable as $\forall R \neg R(x, y)$.

We now show how to recursively assign to each 2-SIB-sentence ϕ a 2S5-sentence ϕ^* with the property that ϕ is 2-SIB-valid if and only if **FullModel** $\rightarrow \phi^*$ is 2S5-valid.

First, we fix an assignment of a propositional variable p_x to each individual variable x , and of a propositional variable p_S to each relational variable S .

DEFINITION 27. Let ϕ be a 2-SIB-formula; then ϕ^* is recursively defined as follows:

1. if ϕ is $S(x, y)$ then ϕ^* is $\exists t[\text{Sngl}(t) \wedge t \subseteq p_S \wedge \exists p, q[\text{Sngl}(p) \wedge \text{Sngl}(q) \wedge R_1(p, p_x) \wedge R_1(q, p_y) \wedge R_2(t, p) \wedge R_2(t, q)]]$.
2. $(\phi_1 \wedge \phi_2)^*$ is $\phi_1^* \wedge \phi_2^*$;
3. $(\neg\phi)^*$ is $\neg\phi^*$;
4. $(\exists x\phi)^*$ is $\exists p_x(\text{Sngl}(p_x) \wedge \phi^*)$
5. $(\exists S\phi)^*$ is $\exists p_S(T_2(p_S) \wedge \phi^*)$.

THEOREM 28. Let X be a set and \mathfrak{M} be a full 3-tiered model representing $[X]^2$; then for any 2-SIB-sentence ϕ , ϕ is true in the structure $(X, \mathcal{P}([X]^2), \in)$ if and only if ϕ^* is valid in \mathfrak{M} .

PROOF. Let \mathfrak{M} represent $[X]^2$. Then there are functions f , g , and h as in Def. 21. Let σ be a second-order assignment of objects and sets of doubletons, as appropriate, to the first- and second-order variables of ϕ . Then there is a related modal assignment σ^* of propositional values in \mathfrak{M} ; σ^* assigns members of $\{\{w\} : w \in g(T_3)\}$ to first-order variables and assigns subsets of T_2 to second-order variables. The correspondence is determined as follows:

$$\begin{aligned} \sigma^*(p_x) &= \{w\}, \text{ where } w \text{ is chosen so that } h(w) = \sigma(x); \\ \sigma^*(p_R) &= \{f^{-1}(\{x, y\}) : \{x, y\} \in \sigma(R)\}. \end{aligned}$$

Now, proceeding by induction on formulas, we show that if

$$\phi(x_1, \dots, x_n; R_1, \dots, R_m)$$

is a 2-SIB-formula then ϕ^* is satisfied by σ^* at every world in \mathfrak{M} if and only if ϕ is satisfied in $[X]^2$ by σ .

Case 1: ϕ is atomic, say $S(x, y)$; then ϕ^* is

$$\begin{aligned} & \exists t(\text{Sngl}(t) \wedge t \subseteq p_S \wedge \\ & \exists p, q(\text{Sngl}(p) \wedge \text{Sngl}(q) \wedge R_1(p, p_x) \wedge R_1(q, p_y) \wedge R_2(t, p) \wedge R_2(t, q))). \end{aligned}$$

Let $\sigma^*(p_x) = \{v_1\}$ and $\sigma^*(p_y) = \{v_2\}$. Suppose σ^* satisfies ϕ^* in \mathfrak{M} ; then there is a world $w \in \sigma^*(p_S)$ such that its two children w_1 and w_2 are R_1 -related to v_1 and v_2 respectively.

Then by Def. 21, $f(w) = \{h(v_1), h(v_2)\}$. Now, $u \in \sigma^*(p_S)$ if and only if $f(u) \in \sigma(S)$, where $u \in T_2$. Therefore, $\{h(v_1), h(v_2)\} \in \sigma(S)$, so σ satisfies ϕ in $[X]^2$.

The converse implication is similar: if σ satisfies $S(x, y)$ in $[X]^2$, then $\{\sigma(x), \sigma(y)\} \in \sigma(S)$. Now, $\sigma^*(p_x) = \{w_1\}$, where $h(w_1) = \sigma(x)$ and $\sigma^*(p_y) = \{w_2\}$, where $h(w_2) = \sigma(y)$. Since $\sigma(x) \neq \sigma(y)$, $w_1 \not\cong w_2$, so by Def. 22 there is a $w \in T_2$ such that $\{w_1, w_2\} = f(w)$. Then $\{w\}$ witnesses the existential claim in $S(x, y)^*$, as desired.

Case 2: ϕ is $\neg\psi$ or $\psi_1 \wedge \psi_2$: the desired conclusion follows immediately from the inductive hypothesis.

Case 3: ϕ is $\exists x\psi$. Then ϕ^* is $\exists p_x(\text{Sngl}(p_x \wedge \psi^*))$. Suppose σ^* satisfies ϕ^* in \mathfrak{M} . Then there is a world w such that $h(w) = \sigma(x)$ and $\sigma^*[\{w\}/p_x]$ satisfies ψ^* . But $\sigma^*[\{w\}/p_x] = \sigma[h(w)/x]^*$ so $\sigma[h(w)/x]^*$ satisfies ψ^* , and by inductive hypothesis, $\sigma[h(w)/x]$ satisfies ψ ; hence, σ satisfies $\exists x\psi$.

Conversely, suppose that σ satisfies $\exists x\psi$, i.e., that for some $a \in X$, $\sigma[a/x]$ satisfies ψ ; by inductive hypothesis, $\sigma[a/x]^*$ satisfies ψ^* . But $\sigma[a/x]^* = \sigma^*[\{h(w)\}/p_x]$, for some w such that $h(w) = a$. Hence, $\sigma^*[\{h(w)\}/p_x]$ satisfies $\text{Sngl}(p_x) \wedge \psi^*$, so that σ^* satisfies $\exists p_x[\text{Sngl}(p_x) \wedge \psi^*]$, as required.

Case 4: ϕ is $\exists S\psi$; then ϕ^* is $\exists p_S[T_2(p_S) \wedge \psi^*]$. Suppose σ^* satisfies ψ^* ; then there is a set q of tier-2 worlds such that $\sigma^*[q/p_S]$ satisfies ψ^* . Let

$$T = \{\{x, y\} : f(\{x, y\}) \in q\};$$

then $\sigma^*[q/p_S] = \sigma[T/S]^*$, and the conclusion follows as before.

Conversely, if σ satisfies $\exists S\psi$ then there is a relation T over X such that $\sigma[T/S]$ satisfies ψ . By taking $q = f[T]$ we have that q is a set of tier-2 worlds and $\sigma[T/S]^* = \sigma^*[q/p_S]$, whence the conclusion follows. \dashv

THEOREM 29. For any 2-SIB-sentence ϕ , ϕ is 2-SIB-valid if and only if $\text{FullModel} \rightarrow \phi^*$ is 2S5-valid.

PROOF. This follows immediately from the previous results. If ϕ is not valid, then it fails over some set $[X]^2$. Let \mathfrak{M} represent $[X]^2$: then FullModel holds in \mathfrak{M} , but ϕ^* fails. Conversely, if $\text{FullModel} \rightarrow \phi^*$ fails over some model \mathfrak{M} , we have that FullModel holds in \mathfrak{M} but ϕ^* fails. In particular, \mathfrak{M} represents some $[X]^2$, and ϕ fails over $[X]^2$. \dashv

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DEPT. OF LOGIC & PHILOSOPHY OF SCIENCE
 UNIVERSITY OF CALIFORNIA
 IRVINE, CA 92697-5100
E-mail: aldo@uci.edu

DEPT. OF PHILOSOPHY
 UNIVERSITY OF MICHIGAN
 ANN ARBOR, MI 48109-1003
E-mail: rich@thomason.org