Logicism, quantifiers, and abstraction

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Abstract

With the aid of a non-standard (but still first-order) cardinality quantifier and an extra-logical operator representing numerical abstraction, this paper presents a formalization of first-order arithmetic, in which numbers are abstracta of the equinumerosity relation, their properties derived from those of the cardinality quantifier and the abstraction operator.

1 Numbers as *abstracta*

Arithmetic is the theory of the natural numbers $0, 1, 2, \ldots$ with which we have all become acquainted in grade school. In an attempt to capture as much of their properties as the limited means at the disposal of the human race will allow, and because of the basic character of arithmetic as the foundation upon which mathematics and science rest, philosophers have been concerned with the proper formalization of the theory of the natural numbers ever since Frege [1879, 1884]. This line of investigation has lead to the Dedekind-Peano axioms that are nowadays standard, as well as to several set-theoretic reductions, and finally to a renewed interest in the Fregean project as championed by the neo-logicist school of Hale and Wright [2001].

Indeed, there are several main approaches to the formalization of arithmetic. According to one view, arithmetic is best regarded as a first-order theory in which numbers are taken as primitive and their properties laid down by means of particular extra-logical axioms. These axioms were first formulated by Dedekind [1888] and Peano [1889], identifying the basic properties of the successor operation on the natural numbers (as well as possibly the properties of addition and multiplication), and postulating an induction schema expressing that any properties of natural numbers that hold of zero and are preserved by the successor operation hold of all the natural numbers. Although Peano Arithmetic (PA), as the theory as come to be known, is sometimes supplemented by a second-order induction principle, it is standardly expressed at the first order. The insistence on a first-order axiomatization is motivated by the desire not to enter the hostile landscape of second-order logic, which is characterized by the failure of such properties as axiomatizability, compactness, Löwenheim-Skolem, etc., familiar from first-order logic.[1] Let us refer to this view as the *axiomatic* approach.

A second approach was championed by Frege (Frege [1884]) and Russell (Whitehead and Russell [1925]), each one of which provided an account of the natural numbers as intimately related to classes of equinumerous concepts, i.e., as equivalence classes under the "having the same cardinality" relation. In Russell's (unramified) theory of types, numbers are classes of concepts (i.e., propositional functions) such that between any two of these concepts there is a relation that is both one-one and onto. As a consequence of the rigid type-theoretic discipline of the theory, numbers are reduplicated at each type higher than 2, a result that makes it impossible to compare cardinalities across types and hardly squares with our intuitions that there is, in fact, just one class of natural numbers. Such undesirable consequences are avoided in Frege's framework by a characteristic recourse to a type-lowering device, i.e., *concept extensions*.[2] Concept extensions derive from the application of a particular abstraction principle (in the sense in which such principles will be characterized in the present paper), whose crucial function is the assignment of first-order objects to concepts in such a way as to respect a given equivalence relation among concepts (in the case at hand — Frege's theory of extensions — such an equivalence between concepts F and G is given by the relation "having the same objects fall under F as fall under G," but other equivalence relations are possible, giving rise to different abstraction principles). Frege's strategy ultimately failed because it was driven by the desire for unattainable generality — the idea that the assignment of extensions to concepts needs to be universal — whereas as we know from Cantor's theorem there are many more concepts than objects,

[1]Logicians are fond of classifying mathematical theories according to their complexity and expressiveness: second-order logic is so far off the scale in this respect that Richard Montague was heard saying, around 1965, that it does not live in any hierarchy, "past, present, or future" (see Enderton [2008]).

[2]A reconstruction of Frege's *extensionalist* — as opposed to *logicist* — program can be found in Antonelli and May [2005]. More about this later.

even when concepts are taken modulo equiextensionality.

Finally, ever since the acceptance of Zermelo-Fränkel set theory as the privileged framework for mathematics, set-theoretic reductions of arithmetic have become standard. Such reductions proceed by identifying particular representatives for Frege's and Russell's equinumerosity classes. These representatives are given in the form of a linearly ordered sequence of sets, having a first element, and with the additional property that any element has only finitely many predecessors in the sequence. Sometimes, a particular n-membered set is selected as representative for the number n (as when the sequences comprises the von Neumann finite ordinals). But this need not always be the case, as with the Zermelo numbers (standardly, but somewhat inappropriately, referred to as "Zermelo numerals"), in which each element is the singleton containing its unique predecessor, so that the representative chosen for the number n does not itself have n members. This further illustrates the point that, quite in general, representatives chosen for equivalence classes relative to some relation R need not themselves be in the field of R (although they often will): all that is required is that the assignment of representatives respect the equivalence relation.

Each of the above-mentioned approaches is wanting in some respect or other. Formalizing arithmetic as a first-order theory, PA, ultimately leaves the nature of numbers unexplained. This is fine as far as it goes. There is certainly something to be said for taking an approach that focuses on the intrinsic *mathematical* and *structural* properties of the natural numbers, the fact that they form an ω-sequence, without being concerned with their ultimate nature. But this approach fails to take into account other, crucial and extrinsic, properties of the natural numbers, first and foremost their *cardinal* properties, as they are put to use in connection with the act of counting.[3] Natural numbers find their primary domain of application in answering questions such as "How many?" It is a desirable feature of any formal account of arithmetic that the cardinal properties of the natural numbers take center stage, along with the structural properties of ω-sequences. It is this emphasis on the *applicability* of arithmetic that is lacking in any account that privileges structural properties of ω-sequences over cardinal ones. In this respect, PA would still need to be supplemented by a separate account of how numbers are used in counting.

Set-theoretic reductions fare somewhat better than PA when it comes

[3]The natural numbers have, besided their cardinal properties, also *ordinal* properties. Although cardinal and ordinal properties are often taken together, we will see later that they are in fact quite distinct.

to their employment in the assignment of cardinal numbers, if anything just because they are embedded into a richer theory allowing for many sorts of maps between sets of different kinds. The details here depend on the particular set-theoretic reduction being adopted (more on this in a moment), but typically one might say that a given set S has n members if and only if there is a bijection between S and the set of natural numbers preceding n in the linear order. (In the case then of the finite von Neumann ordinals, the set of predecessors of n just *is* the natural number n.)

So we do have the beginnings of an account of the cardinal properties of the natural numbers. But again, this account is due more to the theory's being embedded in a rich set-theoretic universe providing the requisite maps than to an explicit consideration of such properties. Moreover, set-theoretic reductions suffer from a problem first pointed out by Benacerraf [1965]. In that classic paper, Benacerraf argues that there is no privileged way to select one particular set-theoretic reduction over any other one. Given that all set-theoretic reductions (of which there are indeed infinitely many) provide for the same intrinsic structural properties of the natural numbers, how is one to to assess the relative merits of, and decide between, say, Zermelo numerals and von Neumann cardinals? Benacerraf's answer is that there is no principled reason to choose between them, and that since the natural numbers can't be *both* Zermelo numerals and von Neumann ordinals (because certain properties, such as $2 \in 4$, hold in one case but not the other), numbers can't be sets at all.

This leaves us with a last option, the Frege-Russell natural numbers. The great advantage of this approach is that the intrinsic mathematical properties of the natural numbers are derived from their cardinal properties, rather than the other way around. Whereas for both the axiomatic approach and set-theoretic reductions cardinal properties require a separate account (more so, as we have seen, in the axiomatic approach), according to the Frege-Russell approach such properties are central to the account of the natural numbers. The essential lines of such an approach, therefore, appear to be intuitively well-motivated and mathematically elegant. Unfortunately, mathematical implementations are rife with problems: Frege's own attempt in [1903] was notoriously inconsistent, and Russell's [1925] imposition of a type discipline, while blocking the inconsistency afflicting Frege's theory, led to reduplications and restrictions that hardly do justice to actual mathematical practice.

Lately, Hale and Wright [2001] have championed a somewhat different, "neo-logicist" approach, which addresses directly the idea that numbers are related to equivalence classes under the equinumerosity relation, dispensing with the whole apparatus of concept extensions.[4] They introduce a theory of numbers based on an operator mapping each concept F to an object x, construed as "the number of F." The mapping is required to satisfy what has come to be known (again, perhaps improperly) as *Hume's Principle*: concepts F and G are mapped to the same object precisely when there is a one-to-one correspondence between the objects that fall under F and the objects that fall under G. As it was known already to Frege, then, the axioms of PA can be derived, within second-order logic, from Hume's Principle. The resulting neo-logicist system is equiconsistent with second-order arithmetic.

There are two main issues with such an approach. The first is both philosophical and conceptual: Hale and Wright rely on the logical character of Hume's Principle in order to characterize their project as continuous with Frege's original logicist views. The extent to which Hume's Principle enjoys a logically or epistemologically privileged status is, however, debatable, in the light of the fact that there are models of arithmetic where Hume's Principle fails.[5]

But even discounting these worries, a main technical obstacle remains, namely the fact that the neo-logicist program is wholly carried out within the framework of second-order logic, with the accompanying failure of those meta-theoretic properties that make first-order logic so attractive. Nonetheless, the Frege-Russell approach appears to be conceptually superior in its characterization of numbers as *abstracta* of the equinumerosity relation, in the way it derives basic mathematical properties from cardinal ones (and the concomitant emphasis on the *applicability* of arithmetic), and in its intuitive motivation. Is there a way to pursue such an approach setting a course clear of both dangers of second-order intractability and the arbitrariness of set-theoretic reductions? Before we answer this question we need to take a closer look at abstraction principles.

2 Varieties of Abstraction

In a tradition that goes back to Frege [1884], but which has recently been revived by Hale and Wright [2001], abstraction principles have been taken to have the general form of Hume's principle, in the form: the number of F = the number of G if and only if there is a one-one correspondence between

[4] Antonelli and May [2005], instead, go in the opposite direction, developing an explicitly *extensionalist* program dispensing with the logical character of arithmetical principles.

[5] A counter-example to the right-to-left direction of Hume's Principle can be found in Boolos [1996], and a counter-example to the left-to-right direction — usually considered the less questionable one — can be found in Antonelli and May [2005].

the F's and the G's. Accordingly, abstraction principles can then be given as follows (see Rosen [2006]):

$$f(a) = f(b) \Longleftrightarrow R_f(a, b).$$

The above formula asserts that the f of a is the same as the f of b if and only if a and b are appropriately related to each other by R_f, where f is a function of the appropriate type and R_f is an equivalence relation between objects of the same type as a and b. Let us refer to such principles as *Equivalence-Governed Functional Expressions* (EGFE's). Clearly Hume's principle is an EGFE, where the function f assigns "numbers" (whatever these might be) to concepts, and the corresponding R_f is the equinumerosity relation between concepts. Another example, also from Frege [1884], concerns the notion of direction: given lines a and b in (say) the Euclidean plane, the direction of a is the same as the direction of b if and only if a and b are parallel to each other. Here again f assigns directions (whatever these might be) to lines, and the corresponding R_f is the parallelism relation.

Two observations are in order here. The first is simply that, in EGFE's, and in the most general sense, the functional expression f is just an assignment of representatives to the equivalence classes induced by R_f. Nothing more is said about these representatives, other than the assignment must respect the equivalence relation. In particular, nothing is implied as to whether the representative $f(a)$ must itself be in the field of R_f, i.e., whether $f(a)$ is itself of the same type as a — and in fact a member of the equivalence class of a (more about this last condition in a moment). Once we realize that abstraction principles only allow for the choice of representatives, any worries about the special ontological status of abstract objects disappear: anything at all — even ordinary objects — can play the role of these *abstracta*, as long as the choice respects the equivalence relation.

The second observation concerns a seemingly hitherto little-noticed fact, i.e., that several options are available as to the *types* of both arguments and values of f. In particular, each of the arguments and the values of f can, independently of one another, be either of the first or of the second-order. Since it is customary to use upper-case letters for second-order variables and lower-case ones for first-order variables, we can use a convenient scheme to lay out the different possibilities. Let us call a functional expression whose arguments are at both at the first order an expression of the xy kind, one whose arguments are both at the second order of the XY kind, and one whose argument are of the second order and whose values are at the first order of the Xy kind (similarly for the somewhat less interesting case of functional expressions of the xY kind).

In standard (neo-)Fregean orthodoxy, Hume's Principle is an EGFE of the Xy kind: the functional f takes second-order arguments (concepts) and returns first-order values (numbers, as a particular kind of objects). Here "concepts" are construed as genuine second-order entities, of which — as we know from Cantor — there are a great many. There are in fact, many more concepts than objects, whence it follows that no functional f of the Xy kind can be injective.

Many philosophically interesting principles can be seen to be, in fact, just particular EGFE's. Perhaps the simplest case is the set-theoretic principle of *extensionality*:

$$x = y \Longleftrightarrow \forall z(x \in x \leftrightarrow z \in y).$$

Here the functional expression f is just the identity function, and the associated equivalence R_f is equiextensionality. The principle is then seen to be an EGFE of the xy kind.

A related — albeit inconsistent — principle is naïve comprehension, an EGFE of the Xy kind:

$$\{x : \phi\} = \{x : \psi\} \Longleftrightarrow \forall x(\phi(x) \leftrightarrow \psi(x))$$

The functional expression f used above is given by the set-formation operator $\{x : \phi\}$, and the associated R_f is the relation holding between ϕ and ψ when the same objects fall under one as they fall under the other. For comprehension to be viewed as an EGFE of the Xy kind, we need first of all to see that set-formation indeed takes concepts as arguments. But this will follow under a very mild assumption, namely closure of concepts under definability. This assumption is obviously satisfied if, as noted, we take concepts to be genuine second-order entities, for then conceptual space (the space of all subsets of the domain that are available as extensions) comprises *all* subsets of the first-order domain, including all definable ones. It is also evident that this particular EGFE runs afoul of the constraint noted above, for then $\{x : \phi\}$ would indeed be an injection of the concepts into the objects.

It is interesting to notice that the *second-order* comprehension principle is not so problematic. The principle amounts merely to the above-mentioned closure of conceptual space under definability: for any definable property $\phi(x)$ there is a concept P having the same extension. The principle is usually stated as a schema, for each ϕ:

$$\exists P \forall x(P(x) \leftrightarrow \phi(x)).$$

In order to re-cast this principle as an explicit EGFE, let us introduce a functional expression C of the XY kind, governed by the equivalence

$$C(\phi) = C(\psi) \Longleftrightarrow \forall x(\phi(x) \leftrightarrow \psi(x)).$$

This is not quite enough to capture second-order comprehension, as it only implies that there is a mapping of definable concepts into primitive ones that respects equiextensionality, without any indication that the primitive concept assigned to ϕ is in fact equiextensional with ϕ.

This suggests that at times further particular restrictions might be needed for certain EGFE's, namely the already-mentioned requirement that the values of the functional expressions fall within the same equivalence class as the corresponding arguments (obviously this makes sense only for EGFE that are *homogeneous*, i.e., those for which arguments and values are of the same type: these are EGFE's that are either of the xy kind or of the XY kind). Accordingly, let us call an EGFE

$$f(a) = f(b) \Longleftrightarrow R_f(a, b).$$

internal if and only if it satisfies the further requirement that $R_f(a, f(a))$, for every a. It is then seen that second-order comprehension can be viewed as an internal EGFE.

Finally, as a further example of an EGFE of the xy kind, let us identify (in true Fregean spirit) *truth values* with particular first-order entities. Let us denote the truth-value true by a constant $\top$ (so that the value false is denoted by $\neg\top$), and consider an assignment V of truth values to sentences satisfying the EGFE:

$$V(\phi) = V(\psi) \Longleftrightarrow (\phi \leftrightarrow \psi),$$

with V providing the functional expression and material equivalence the corresponding relation. Then, given that $V(\top) = \mathsf{true}$, we have

$$V(\phi) = \mathsf{true} \Longleftrightarrow (\phi \leftrightarrow \top),$$

whence (by propositional logic)

$$V(\phi) = \mathsf{true} \Longleftrightarrow \phi,$$

and we have recovered Tarski's T-biconditionals.[6]

[6] An *internal* version can be obtained by assuming V to take sentences into sentences, specifically into $\{\top, \neg\top\}$ (as opposed to truth values as extra-linguistic objects). Then we still recover the Tarski biconditionals, and moreover $V(\phi) \leftrightarrow \phi$, so that the corresponding EGFE is perforce internal.

Let us now take a quick look at examples of EGFE's of the xY kind. We can see that Frege's original example concerning direction can be viewed in this way: given lines a and b, the direction of a equals the direction of b if and only if a and b are parallel. Lines are here construed as *objectual*, whereas the *direction* assigned to a line a is construed as the *concept* under which all and only the lines parallel to a fall.[7]

Another example deals with *individual concepts*, i.e., those concepts under which one and only one objects falls (these are sometimes taken to convey the "essence", or perhaps the "individual essence," of that object — a construal we need not be concerned with here). If we take the relevant xY operator to be denoted by IC, the corresponding EGFE will be

$$IC(a) = IC(b) \Longleftrightarrow a = b.$$

As it is particularly clear from this last example, EGFE's of the xY kind tend to be uncontroversial, at least to the extent that the injection from objects to concepts does not carry the "inflationary" thrust of the reverse injection from concepts into objects.

As we have seen, a great many EGFE's are put to use in a variety of philosophical contexts to accomplish a number of tasks. Not all these qualify as *abstraction principles*. Indeed, it would seem that abstraction properly deals with concepts, and so in some sense "lives" in conceptual space. In one of the most ordinary uses of the word, "abstraction" is employed to refer to the process of concept formation, as it occurs, e.g., in children or in science (whereby a concept is "abstracted" from a variety of instances). In the present context, however, abstraction properly understood deals with the assignment of entities of one kind or another to concepts. Accordingly, we distinguish between *second-order abstraction*, as expressed by EGFE of the XY kind, and *first-order abstraction*, as expressed by EGFE's of the Xy kind. The former is technically important (e.g., for its interaction with second-order induction principles), but philosophically less threatening, at least on the understanding of concepts as genuine second-order entities.[8] The latter, on the other hand, gives us a mapping of concepts into the objects and at times teeters on the brink of inconsistency. We will henceforth refer to first-order abstraction as abstraction *simpliciter*, and make it the object of our inquiry.

[7] Of course, this need not be the only available interpretation of this principle.

[8] But as a counterpoint see Antonelli and May [2005] where numbers are identified with particular *concepts* (rather than objects) and Hume's principle is accordingly re-cast as an EGFE of the XY kind.

Having agreed on this terminological point, we need to consider the logical status of abstraction principles. Neo-logicists such as Hale and Wright [2001] have made a great deal of the privileged status of such principles, and especially HP. They claim that HP enjoys a status not unlike that of logical truth, and that HP is somehow "constitutive" of the notion of number. They are undeterred that principles very much like HP, *viz.*, naïve comprehension, turn out to be inconsistent (the so-called "Bad Company" objection), perhaps making consistency itself the hallmark of acceptability for abstraction principles. But this line of argument has been further criticized by R. G. Heck [1992] and Weir [2003], pointing out that there are individually consistent but pairwise incompatible abstraction principles: if both principles in such a pair are acceptable, hence at least to an extent analytic, then both need to be regarded as true, which is of course impossible (this is the so-called "Embarassment of Riches" objection of Weir [2003]). Finally, Boolos [1998, p. 231] notes that R. G. Heck [1992] (where such pairs of principles where first introduced) appears "to do in, once and for all, the idea that 'contextual definitions' like Hume's principle or Basic Law V, have, in general, any privileged logical status."

Once we accept Boolos' conclusion that there is nothing special about the logical status of abstraction principles, that there is nothing "logical" or "analytic" about them, we are free to take a detached look, notice that some such principles are good and some bad, that in general they can be put to use in a variety of philosophical contexts, and that they are capable of accomplishing a variety of tasks. Even the "inflationary" status of HP, for instance, can be viewed as a tool, to be used for certain purposes, but without any claim to privileged status.

But is what is then left of *logicism*? What are the prospects of the ambitious program initiated by Frege and revived by the neo-logicist school?

3 Logicism

Frege's original program aims to combine two largely incompatible views: *logicism*, construed as a reduction of arithmetic to (higher-order) logic; and *extensionalism*, construed as a theory of concept extensions as abstract objects. This terminology is drawn from Antonelli and May [2005]; Dummett [1991, p. 301] makes the same point, referring to the latter as "platonism" — Frege's view that there are logical objects in the form of concept extensions — and notices that the two are not only independent, but in fact are in tension, a fact that results in the contradiction uncovered by Russell. Hence, it appears that, for the sake not only of consistency but also of conceptual purity, the two halves of Frege's program are best pursued independently of each other. Antonelli and May [2005] pursue extensionalism, by providing a theory in which concept extensions are explicitly governed by extra-logical principles, and arithmetic recovered as a second-order theory identifying numbers with particular concepts. The other option is to pursue logicism for its own sake.

Logicism is most often characterized as the view that "mathematics *is* logic." Vague as that may be, the slogan has been usually articulated in a *reductionist* fashion by identifying some principle, claimed to enjoy some logically or epistemologically privileged status, to which arithmetic turns out to be proof-theoretically reducible. Such principle was identified by Frege in Basic Law V, and by Hale and Wright [2001] in Hume's Principle. But these kinds of abstraction principles still carry the imprint of Frege's reliance on logical objects (full-blown, of course, only in the case of Basic Law V, but still evident nonetheless in the case of Hume's Principle). But as pointed out by [Dummett, 1991, p. 302], on the "natural view" of logic, *there are no logical objects*: all that is needed or required for the completion of the logicist program is an interpretation of all mathematical statements (or at least the arithmetical ones) into a logical language. The language standardly chosen for this task is second- or higher-order logic, as in Whitehead's and Russell's *Principia Mathematica*. That attempt again floundered as a realization of logicism because of the failure to justify the existence of abstract objects on purely logical grounds, where the existence of such objects was required for a faithful interpretation of arithmetic into logic.

All of these attempts — including Dummett's own characterization — rely on a reductionist construal of logicism: the idea that the logicist slogan that mathematics is logic is best understood as requiring either a proof-theoretic reduction or a semantic interpretation of arithmetic into logic. Reductionism enjoyed a certain currency in the philosophy of science beginning with the work of Nagel [1961], who championed inter-theoretic reduction via "bridge principles" playing a role not too dissimilar from that of Basic Law V or Hume's Principle in the (neo-)logicist attempts. But even if reductionism can be defended in the case of empirical science (whether successfully or not is a matter for discussion), it appears somewhat implausible when applied to arithmetic. In such a case this reductionist construal is not the only, the most general, or indeed the most natural interpretation of logicism.

On the most natural interpretation, logicism takes seriously Dummett's remark that, for Frege "the notion of cardinal number is *already* a logical one, and does not need a definition in terms of [concept extensions, or other logical objects] to make it so" (Dummett [1991, p. 224]). Taken at face value, this remark suggests an alternative interpretation of logicism in which cardinality is taken to be a genuine logical notion *per se*, and as such it is employed as one of the basic or indeed the fundamental building block in designing a formal framework adequate for the representation of arithmetical facts.[9] We decide to pursue this particularly general, useful, and novel construal, which can be summed up in the slogan:

> Cardinality is a logical notion.

Accordingly, we proceed to develop an account of arithmetic in which the logicist thrust is carried by the logical framework rather than by abstraction principles. Cardinality itself will play a central role as a primitive logical notion. And what is more suited for the realization of logical notions than the very notion of a *quantifier*?

4 The modern view of quantifiers

Logicians have a very specific view of quantifiers, which is passed along to generation after generation of students in introductory logic courses. A quantifier is any *expression* that can be used in conjunction with a *variable* and an *open formula* to produce a *sentence*. Students are then told that of these quantifiers there are at most two: the existential quantifier $\exists$ and the universal quantifier $\forall$ that combine with variables such as x and y to and formulas $\varphi(x)$ or $\varphi(y)$ to produce sentences $\exists x\varphi(x)$ or $\forall y\varphi(y)$, intuitively saying that something (in some intended domain of discourse) is a φ, or that everything in that same domain is φ. In fact, students are not told *what* a quantifier is, nor is it explained to them why it happens that there are only two of these quantifiers (or perhaps one, if the other one is viewed as definable).

The modern view of quantifiers is much more complex than this. The study of *generalized quantifiers* (as such a view has come to be known) initiates with the work of Mostowski [1957] and continues with that of Montague [1974]. This work on generalized quantifiers spans linguistics and mathematical logic, the linguists focusing on quantifiers as tools for natural language semantics, and the logicians focusing on the expressive power and properties such as axiomatizability, decidability etc.

[9] It should be noted that Dummett himself does not go on to develop the possibility of non-reductionist interpretations of logicism, thus failing to heed his own characterization.

Interestingly, in fact, the study of generalized quantifiers can be traced back to the work of Frege, and specifically §21 of *Grundgesetze der Arithmetik* where Frege asks us to consider the forms:

$$\top\!\!-\!\overset{a}{\smile}\!-\!\!\top\; a^2 = 4 \quad \text{and} \quad \top\!\!-\!\overset{a}{\smile}\!-\!\!\top\; a > 0.$$

These forms of the "conceptual notation" correspond to the modern formulas $\exists a(a^2 = 4)$ and $\exists a(a > 0)$. Frege notices that these forms can be obtained from $\top\!\!-\!\overset{a}{\smile}\!-\!\!\top\; \phi(a)$ by replacing the function-name placeholder $\phi(\xi)$ by names for the functions $\xi^2 = 4$ and $\xi > 0$ (and the placeholder cannot be replaced by names of objects or of functions of 2 arguments). The two functions just mentioned take numbers as arguments and return the value true if those numbers are square roots of 2 or (respectively) positive, and false otherwise. Since the functions take objects (in this case, numbers) as arguments they are referred to as *first-level* functions. Frege refers to first-level functions that return the values false and false as *concepts*.

So the above forms can be regarded as values of the same function for different arguments. Now, these arguments, as we just noted, are themselves functions. It follows that the form

$$\top\!\!-\!\overset{a}{\smile}\!-\!\!\top\; \phi(a)$$

is a *second-level function*, because its arguments are *first-level functions*. Since the arguments of the form $\top\!\!-\!\overset{a}{\smile}\!-\!\!\top\; \phi(a)$ are functions that return truth values (i.e., concepts) the same holds for the form $\top\!\!-\!\overset{a}{\smile}\!-\!\!\top\; \phi(a)$. Such a form — a quantifier — is therefore a *second-level concept*. This is not a mere coincidence, but a general fact: from a Fregean point of view, quantifiers are second-level concepts.

This view of quantifiers as higher-level entities can be found again in the modern approach, which takes a characteristically general stance on the matter. Given a *domain of discourse* (i.e., a non-empty set) D, a quantifier Q over D, then, is just a collection of subsets of D: $\mathsf{Q} \subseteq \mathscr{P}(D)$.

This account is consistent with the traditional view of quantifiers as operators from formulas to sentences. Let us understand a formula ϕ in one free variable x as denoting a subset of D *viz.*, the collection $[\![\phi]\!]$ of those $d \in D$ that satisfy the formula ($[\![\phi]\!]$ can be thought of as the *extension* of ϕ in D). Then a quantifier can, in turn, be identified with a collection of subsets of D. It is easier to understand what this view amounts to by looking at some examples.

1. The ordinary universal quantifier $\forall$ can be identified with the collection of subsets of D that contains D itself as its only member: $\forall = \{D\}$; a

sentence of the form $\forall x\varphi(x)$ is true over D precisely when every $d \in D$ satisfies φ, i.e., when the extension of $\varphi(x)$ over D is D itself. Hence, $\forall$ can be identified, semantically, with $\{D\}$.

2. Similarly (and dually) the ordinary existential quantifier $\exists$ can be identified with the collection of all non-empty subsets of D, i.e., $\exists = \{X \subseteq D : X \neq \emptyset\}$; a sentence of the form $\exists x\varphi(x)$ is true precisely when some $d \in D$ satisfies $\varphi(x)$, i.e., the extension of φ over D is non-empty.

3. Once we adopt the more abstract viewpoint of generalized quantifiers, it is natural to consider quantifiers beyond the usual ones, e.g., the quantfier "there exist exactly k," usually written $\exists!^k$. This quantifier can be identified with the collection of all k-membered subsets of D, i.e., $\exists!^k = \{X \subseteq D : |X| = k\}$; then $\exists!^k x\varphi(x)$ is true precisely when there are exactly k objects in D that satisfy φ.

4. According to a view first proposed by Montague, proper names can also be identified with collections of subsets of D, and therefore they can be viewed as generalized quantifiers as well. A sentence such as "John runs" is true precisely when John belongs to the collection of all the running objects in D; accordingly, we identify the quantifier "John" with the collection of all subsets of D containing John as a member: $\mathsf{John} = \{X \subseteq D : \text{John} \in X\}$.

There are also extreme examples, which reduce to triviality. We could, for instance, consider the *empty* first order quantifier, $\mathsf{Q}_\emptyset$, i.e., the empty collection of subsets of D. Then we have that $\mathsf{Q}_\emptyset x\phi(x)$ is true precisely when $\{x \in D : \phi(x)\} \in \mathsf{Q}_\emptyset$, i.e., *never*. $\mathsf{Q}_\emptyset x\phi(x)$ is an identically false sentence for any ϕ. Similarly, we could consider the quantifier $\mathsf{U} = \mathscr{P}(D)$ such that $\mathsf{U}x\phi(x)$ is identically true for any ϕ.

All the examples of quantifiers we have seen so far apply to a single open formula $\phi(x)$ at a time: they are, as we will say, *unary*.[10] But in fact, some quantifier are not only best viewed as applying to more than one such formula, but they are such that no other interpretation is possible. Consider the following examples:

1. All A are B: $\mathsf{All} = \{(A, B) : A \subseteq B\}$

2. Some A are B: $\mathsf{Some} = \{(A, B) : A \cap B \neq \emptyset\}$

[10] They are also referred to as *monadic*, but we take monadic quantifiers to apply to formulas having only one free variable, dyadic quantifiers to apply to formulas with two free variables etc. In this paper we only consider monadic quantifiers.

3. Most A are B: $\mathsf{Most} = \{(A, B) : |A \cap B| > |A - B|\}$;

4. Twice as many A as B are C:

$$\mathsf{Twice} = \{(A, B, C) : |A \cap C| = 2 \cdot |B \cap C|\}.$$

Here, as is well known, the first two quantifiers All and Some can be represented by means of *unary* quantifiers applied to Boolean combinations of their arguments. For instance, where, as before, $\forall = \{D\}$, we have

$$\mathsf{All}(A, B) = \forall((D \setminus A) \cup B),$$

and similarly:

$$\mathsf{Some}(A, B) = \exists(A \cap B).$$

However, not all binary quantifiers can be represented in this form, i.e., as a unary quantifier applied to a Boolean combination of their arguments. One example is Most. There is no Boolean term $F(X, Y)$ such that Most is a subset of $\{F(A, B) : A, B \subseteq D\}$ (a binary Boolean term in X and Y is a combination of X and Y by means of a finite number of applications of union, intersection and complementation; a binary Boolean term clearly maps $\mathscr{P}(D)^2$ in to $\mathscr{P}(D)$).

All the above quantifiers are *first-order*. This is well known. It is important, however, to characterize this notion precisely in semantic terms: a binary quantifier is "first-order" if and only if it is a subset of $\mathscr{P}(D) \times \mathscr{P}(D)$, i.e., if it expresses a relation between subsets of D. And similarly, a unary quantifier is first-order if and only if it is a collection of subsets of D. According to this definition, some quantifiers are called first-order even if they are not definable by a first-order formula. Let us make this second idea precise as follows.

Let $\mathscr{L}$ be a standard first-order language, with at least the identity predicate $=$, and possibly other extra-logical constants. We say that a binary quantifier $\mathsf{Q}\,(A, B)$ is *first-order definable* in $\mathscr{L}$ over D if and only if there is a formula $\phi \in \mathscr{L}(P, Q)$ such that

$$\mathsf{Q}\,(A, B) \iff \langle D, A, B\rangle \models \phi(P, Q);$$

i.e., we consider the (expanded) language $\mathscr{L}(P, Q)$ containing two extra 1-place predicate symbols P and Q, and correspondingly obtain a structure for $\mathscr{L}(P, Q)$ by taking D as domain and interpreting P by A and Q by B. If Q is first-order definable, then there is formula ϕ in $\mathscr{L}(P, Q)$ satisfying the above equivalence.

According to this account, the unary monadic quantifier $\mathsf{at\ least\ two}$, denoted by $\mathsf{Q}_{\geq 2}$, is first-order definable in the pure language of identity over D, for:

$$\mathsf{Q}_{\geq 2}(A) \Longleftrightarrow \langle D, A\rangle \models \exists x \exists y (x \neq y \wedge Px \wedge Py)$$

There are quantifiers that, although first-order from a purely semantic point of view, nonetheless exceed the bounds of first-order logic as ordinarily conceived, in that they are not first-order definable in the sense just given above. For instance, Most is first-order but not first-order definable.

It helps clarify the matter further to compare the above with the case of genuine *second-order* quantifiers. Second order quantifiers are just collections of (or, more generally, relations among) first-order quantifiers. If Q is a first-order quantifier, then the sentence $\mathsf{Q}\, x\, \phi(x)$ is true if and only if $\{x \in D : \phi(x)\} \in \mathsf{Q}$. Analogously, if Q is second-order, then the sentence $\mathsf{Q}\, P\, \phi(P)$ is true if and only if $\{P \in \mathscr{P}(D) : \phi(P)\} \in \mathsf{Q}$. It follows that whereas first-order quantifiers are collections of subsets of D, second-order quantifiers are collections of collections of subsets of D, i.e., collections of first-order quantifiers.

We have seen that (using superscripts to identify the order) the first-order existential quantifier $\exists^1$ is identified with the collection of all non-empty subsets of D. Similarly, the second-order existential quantifier $\exists^2$ is identified with the collection of all non-empty collections of subsets of D, that is, with the collection of all non-empty first-order quantifiers: in other word, with the collection of all first-order quantifiers other than $\mathsf{Q}_{\emptyset}$. To make things clearer, consider the sentence

$$\exists^2 P \forall^1 x Px$$

which says that there exists a universal predicate over D. Then we have:

$$\begin{aligned} \exists^2 P \forall^1 x Px &\Longleftrightarrow \{P \in \mathscr{P}(D) : \forall^1 x Px\} \in \exists^2 \\ &\Longleftrightarrow \{D\} \in \exists^2 \\ &\Longleftrightarrow \forall \in \exists^2. \end{aligned}$$

Thus, in general, $\exists^2$ can be identified with $\{X \in \mathscr{P}^2(D) : X \neq \emptyset\}$.

The distinction is between first- and second-order quantifiers is *semantical*, not merely notational. This point goes hand-in-hand with the previous one, that there are first-order quantifiers that exceed the boundary of expressibility in first-order logic. But just because a quantifier is not definable in ordinary first-order languages does not mean that it is second-order.

A property that plays a crucial role in the modern conception of quantifiers is the following, where $\mathsf{Q}\,(A,B)$ is binary first-order:

- *Permutation invariance*: if π is a permutation of D, then $\mathsf{Q}\,(A,B)$ holds iff $\mathsf{Q}\,(\pi[A],\pi[B])$ holds, where $\pi[X]$ (for $X \subseteq D$) is the pointwise image of X under π: $\pi[X] = \{\pi(y) : y \in X\}$.

The reason this property plays such a pre-eminent role is that there is a long tradition, traceable back at least to the work of Alfred Tarski, according to which being invariant under permutations is the hallmark of *logicality.* Logical notions deal with questions that apply to objects in the domain irrespective of their specific nature. Quantifiers are logical notions because they answer the question "How many?" with no concern for the specific nature of the objects in question. Hence, the answer should be unaffected by permutations of those objects.

5 Standard Cardinality Quantifiers

Based on the previous discussion, among the quantifiers that are of especial interest are those that directly deal with *cardinality* notions. We single out two closely related binary quantifiers of this kind:

- The *Härtig quantifier*: $\mathsf{I}(A,B) \Longleftrightarrow |A| = |B|$;
- the *Rescher quantifier*: $\mathsf{R}(A,B) \Longleftrightarrow |A| > |B|$.

The Härtig quantifier holds of subsets A and B of D precisely when there are exactly as many A's as there are B's, whereas the Rescher quantifier holds of subsets A and B precisely when there are (strictly) more A's than B's. These quantifiers, first introduced by Rescher [1962] and Härtig [1965], have been extensively studied from a mathematical point of view. Their defining feature is that they deal with cardinality notions *directly*, without appealing to any separately given mathematical machinery.

Compare this to the situation in set theory, where in order to express certain relationship between the cardinality of two given sets, one has to appeal to the existence of certain *other objects* in the domain of quantification — such objects are, in turn, sets of a certain kind, containing ordered pairs as members and satisfying certain further conditions. Alternatively, one can express such cardinality notions at the second order, by asserting the existence of *relations* satisfying certain further constraints. These are not by any means the only ways in which one can go about talking about cardinality. Alternatively — as we do here — one can take cardinality notions as *linguistic primitives* and explore the expressive power of the resulting linguistic framework. A first attempt in this direction will lead towards quantifiers such as the two mentioned above.

Notice that Härtig's quantifier is definable from Rescher's, although only using the axiom of choice in essential way: $\mathsf{I}(A, B)$ holds if and only if both $\neg\, \mathsf{R}(B, A)$ and $\neg\, \mathsf{R}(A, B)$ hold.[11] The converse is not true: Rescher's quantifier cannot be defined from Härtig's. Moreover, both quantifiers are *semantically* first order, in that both express binary relations between subsets of the domain. It is true that the relations they express can be formulated at the second order, as depending on the existence of certain higher-order objects, *viz.*, functions of a certain kind, but this fact has no bearing on the semantical nature of the notions represented by the quantifiers. The argument that I and R are higher-order because they express notions that *can* be represented at the second order carries no more force than the parallel argument that they are, in fact, first-order because those notions can also be represented at the first-order (e.g., in a suitable set theory). All that matters, in assessing the nature of these (or any other) quantifiers are the semantical facts.

Both quantifiers are permutation-invariant: If π is a permutation of D, then clearly there are just as many A's as B's if and only if there are just as many π-of-As as there are π-of-Bs; i.e., $|A| = |B|$ if and only if $|\pi[A]| = |\pi[B]|$ (and similarly for Rescher's quantifier). This fact lends additional support to the view that cardinality quantifiers such as I or R express genuine logical notions.

In keeping with this view, we consider a language in which cardinality quantifiers are taken as *primitive logical machinery*, and explore their properties in conjunction with the barest logical apparatus. The idea of the logical character of cardinality notions is not, as we know, new. It is, to an extent, by sheer historical accident that Frege decided to develop his *Begriffschrift* taking the (precursors of) the standard quantifiers $\exists$ and $\forall$ as primitive.[12] Another option that was in principle available to him through the work of Dedekind was to take the notion of *function* as primitive instead, and had he done that, perhaps the field would look quite a bit different. More recently, impetus to re-evaluate the logical and epistemological

[11] The axiom of choice, in the form of the trychotomy principle, is needed in order to go from $|A| \not> |B|$ to $|A| \leq |B|$.

[12] Indeed, a case can be made that Frege did not take a unary quantifier such as $\exists$ or $\forall$ as primitive, but rather a version of the binary quantifier $\mathsf{All}(A, B)$.

character of cardinality notions has come through the neo-logicist work of Hale and Wright [2001]. The present approach, however, takes this idea a lot further and much more seriously than Hale and Wright ever did.

A point is worth making here. The study of generalized quantifiers is always carried out taking first-order logic for granted. Whenever logicians and linguists are interested in the properties of some quantifier Q, they explore the expressiveness of the language $\mathscr{L}(\mathsf{Q})$ obtained by adding Q to full-fledged first-order logic (see Peters and Westerståhl [2006], for instance). In what follows, instead, and in keeping with the intuition of the logical character of cardinality notions, we take cardinality quantifiers as the only quantifiers in the language, and explore the expressive properties of the resulting logical framework.

In order to avoid an implicit appeal to the axiom of choice in defining equinumerosity, we will focus on a quantifier F (referred to as the "Frege quantifier") that holds between subsets A and B of the domain if and only if the cardinality of A is less than or equal to that of B, i.e., if and only if there is an injection from A to B.[13] Formally, we consider a *first-order* language $\mathscr{L}$ with formulas built up from (individual or predicate) constants by means of Boolean connectives ($\wedge$, $\vee$, $\neg$, and $\rightarrow$) and the quantifier $\mathsf{F}\,x$ satisfying the clause:

> if $\phi(x)$, $\psi(x)$ are formulas and x a variable, then $\mathsf{F}\,x(\phi(x), \psi(x))$ is a formula.

$\mathsf{F}\,x$ is then a binary quantifier (like All). As for the Rescher quantifier, we abbreviate $Fx(\phi, \psi) \wedge \mathsf{F}\,x(\psi, \phi)$ by $\mathsf{I}x(\phi, \psi)$.

The Frege quantifier can be given a standard interpretation by singling out a class of models and laying down truth (in fact, satisfaction) clauses for the language. As for standard first-order logic, a model $\mathfrak{M}$ with non-empty domain D provides an interpretation for the non-logical constants of $\mathscr{L}$ in the usual way (e.g., n-place predicates are mapped onto relations $\subseteq D^n$, etc.)

Given a formula $\phi(\overline{x})$ and a function s assigning objects from D to the variables of $\mathscr{L}$, satisfaction $\mathfrak{M} \models \phi[s]$ is also defined in the usual way, but with the additional clause:

$$\mathfrak{M} \models \mathsf{F}\,x(\phi, \psi)[s] \iff \exists f : \{s(x) : \mathfrak{M} \models \phi[s]\} \xrightarrow{1-1} \{s(x) : \mathfrak{M} \models \psi[s]\}.$$

Alternatively, if $s\frac{\overline{a}}{\overline{x}}$ is an assignment just like s, except "shifted" to assign $\overline{a} = a_1, \ldots, a_k$ to $\overline{x} = x_1, \ldots, x_k$ (respectively), we can define the extension

[13] Then $|A| = |B|$ follows from $|A| \leq |B|$ and $|B| \leq |A|$ just by the Schröder-Bernstein theorem, without appeal to the axiom of choice.

of ϕ in $\mathfrak{M}$, relative to s as:

$$[\![\phi]\!]_s^{\overline{x}} = \{\overline{a} : \mathfrak{M} \models \phi[s\tfrac{\overline{a}}{\overline{x}}]\};$$

then the above clause becomes:

$$\mathfrak{M} \models \mathsf{F}\, x(\phi, \psi)[s] \leftrightarrow \exists f : [\![\phi]\!]_s^x \xrightarrow{1-1} [\![\psi]\!]_s^x.$$

We have thus defined a completely rigorous semantics for the language $\mathscr{L}_{\mathsf{F}}$, comprising just the Frege quantifier along with connectives and non-logical constants. As a first step in using our newly-found language, we notice that the standard first order quantifiers are expressible in $\mathscr{L}_{\mathsf{F}}$:

- $\forall x \phi(x) = \mathsf{F}\, x(\neg\phi(x), x \neq x)$;
- $\exists x \phi(x) = \neg\mathsf{F}\, x(\phi(x), x \neq x)$.

The first formula expresses the fact that everything is ϕ if and only if there in an injection of the complement of ϕ into the empty set, i.e., if and only if the complement of ϕ is itself empty, and thus if and only if everything in D falls within the extension of ϕ. Dually, the second formula expresses that something is ϕ if and only if there is no injection of ϕ into the empty set. Hence in what follows we will help ourselves to the abbreviations $\forall x$ and $\exists x$. But the language turns out to be much more expressive than ordinary first-order logic. While infinity cannot be characterized using only $\forall$ and $\exists$, there is an *axiom of infinity* in the pure identity fragment of $\mathscr{L}_{\mathsf{F}}$:

AxInf: $$\exists y \mathsf{F}\, x(x = x, x \neq y),$$

which asserts that there is an injection of D into a proper subset of itself, so that D is Dedekind-infinite. AxInf is then true in *all and only* the infinite models, and therefore, its negation is true in all and only the finite models. This fact shows that the Frege quantifier, while still first-order, far outstrips the expressive resources of ordinary first-order logic, in which, as is well known, finitess cannot be adequately represented. As a consequence, *compactness fails* in $\mathscr{L}_{\mathsf{F}}$.

Let us abbreviate by $\mathsf{Fin}\, x\, \phi(x)$ the statement that $\{x : \phi(x)\}$ is Dedekind finite: $\forall y \neg\mathsf{F}\, x(\phi(x), \phi(x) \wedge x \neq y)$. As before, this statement completely captures the fact that extension of ϕ, i.e., $[\![\phi(x)]\!] = \{a \in D : \mathfrak{M}, a \models \phi(x)\}$ is a finite set. Using this device, it's easy to see that there is a sentence ϕ of the language $\mathscr{L}_{\mathsf{F}}(<)$ comprising one binary predicate symbol $<$ as a non-logical constant that is true if and only if the interpretation of $<$ in $\mathfrak{M}$ is a relation having order type $\leq \omega$. Using such a sentence it is then possible to characterize "true" arithmetic, i.e., the set of all sentences that are true in the standard model. The sentence is obtained as the conjunction of the following three clauses:

- $<$ is a strict transitive linear order; and
- $\exists x \forall y (y \neq x \rightarrow x < y)$; and
- $\forall x \,\mathsf{Fin}\, y (y < x)$.

The last two clauses express that the interpretation of ϕ has a first element and that each element of the domain has only finitely many predecessors in the order denoted by $<$. Then taking the conjunction of such a sentence ϕ with the axiom of infinity AxInf we obtain a sentence θ that is true in a model precisely if (the interpretation of) $<$ is a countably infinite linear order. Finally, conjoining this last sentence θ with a set of arithmetical axioms for addition and multiplication (such as, for instance, PA minus induction) we are able to characterize the standard model $(\mathbb{N}, +, \times)$ up to isomorphism.

The language comprising the Frege quantifier F is then quite expressive indeed. In fact, the characterization of "true" arithmetic just noted gives, beside a rather spectacular failure of the Löwenheim-Skolem theorem, also that the set of $\mathscr{L}_{\mathsf{F}}(+, \times)$ validities is not recursively axiomatizable.[14]

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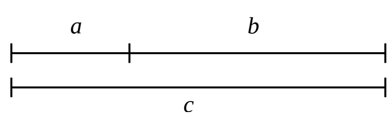
[14] But note that, perhaps surprisingly, the set of $\mathscr{L}_{\mathsf{I}}(<)$ validities *is* decidable. In general, results about the expressive power of cardinality quantifiers interpreted as above are well-known — although perhaps undeservedly forgotten — in the case of the Härtig quantifier, which was extensively studied in the 1970's. I independently re-discovered some of them when I first started looking at the Frege quantifier.

As further evidence of the expressive power of F, consider the following. It is well known that addition is not definable in first-order logic over the structure $(\mathbb{N}, <)$. However, the following defines addition in $\mathscr{L}_{\mathsf{F}}$:

$$a + b = c \iff (\mathbb{N}, <) \models \mathsf{I} x (x < b, a \leq x < c).$$

The proof, "without words" (in the style of Nelsen [1993]), only requires the diagram:



6 The general interpretation of the Frege quantifier

As is well known, second-order quantifiers can be given, beside a *standard* interpretation, also a so-called *general* interpretation (first introduced by Henkin [1950]). On such a general interpretation second-order quantifiers such as $\forall P$ or $\exists P$ are taken to range not over the "true" powerset of D (or of D^n, in case P is an n-place predicate symbol), but over some previously given universe of subsets of D. So while standard models for second-order logic are indistinguishable from first-order models, general models carry, beside a domain D, also a universe of n-place relations over D (for each n). In practice, such a universe of relations will satisfy some closure conditions — it will be, e.g., closed under *definability*, thereby satisfying the second-order comprehension axiom (an EGFE of the XY kind, as we saw).

Surprisingly, *first-order* quantifiers can also be so interpreted, a fact that — to my knowldge — has gone apparently hitherto unnoticed. Perhaps the simplest example is the general interpretation of the ordinary first-order existential quantifier $\exists$. As we have seen, the ordinary existential quantifier ranges over the collection of *all* non-empty subsets of D. It is then natural to consider the existential quantifier $\exists^*$ that ranges over *some* collection of non-empty subsets of D. Dually, one can also consider the general universal quantifier $\forall^*$, ranging over a collection of subsets of D containing D itself as a member. The question of whether the logic of such a quantifier can be axiomatized has a somewhat unexpected answer, but the reader is referred directly to Antonelli [2007] so as not to spoil the surprise.

In the present context, however, we are interested in the general interpretation of the Frege quantifier F. Under the standard interpretation, as we have seen, such a quantifier is quite expressive (see Herre et al. [1991] for a survey of results concerning the Härtig quantifier[15]). Those results apply, *mutatis mutandis*, to the Frege quantifier F. This provides the motivation to focus on a less extravagant interpretation of F.

In order to specify the general interpretation of F we need to single out a class of models, which will, in turn, determine a class of valid sentences — a *logic*. Accordingly, by a *general model* for $\mathscr{L}_{\mathsf{F}}$ we understand a structure $\mathfrak{M}$ providing a non-empty domain D and interpretations for the non-logical constants, as well as a collection $\mathcal{F}$ of 1-1 functions $f : A \to B$ with $\mathrm{dom}(f) = A$, and $\mathrm{rng}(f) \subseteq B$, for $A, B \subseteq D$. The corresponding

[15] The precise expressive power if I depends, at some point, on whether the Continuum hypothesis is true or not.

satisfaction clause for the quantifier F then has the form:

$$\mathfrak{M} \models \mathsf{F}\, x(\phi, \psi)[s] \iff (\exists f \in \mathcal{F})\, f : [\![\phi]\!]_s^x \xrightarrow{1-1} [\![\psi]\!]_s^x.$$

The satisfaction clauses for atomic formulas and Boolean combinations of formulas are as ususal. As is often the case for general interpretations, we further constrain $\mathcal{F}$ to satisfy certain *closure conditions*:

- For each A, the identity map on A belongs to $\mathcal{F}$ (including the empty map on $\emptyset$);

- if $f \in \mathcal{F}$ and $f : A \to B$ and $x \notin A$ and $y \notin B$, then there is a $g \in \mathcal{F}$ such that $g : A \cup \{x\} \to B \cup \{y\}$.

A further closure condition that might be adopted is the more general closure of $\mathcal{F}$ under definability, but in order to keep the basic semantic framework as simple as possible, we choose to impose such closure by laying down a specific axiom to that effect; such an axiom will be given later.

On the general interpretation of F, the language $\mathscr{L}_{\mathsf{F}}$ shares many of the properties of first-order logic. There are at least two ways to arrive at this result. On the first strategy, one identifies a recursive (or recursively enumerable) set of axioms characterizing the general models for $\mathscr{L}_{\mathsf{F}}$, and proceeds to prove a completeness theorem. There is, however, a more direct route, which is all that is required for the purposes of the present paper.[16]

The second strategy more simply amounts to a direct interpretation of $\mathscr{L}_{\mathsf{F}}$ into a suitable first-order theory. In order to give such an interpretation, we introduce a function symbol $f_{\phi,\psi}$ for each pair $\phi(x)$, $\psi(x)$ of formulas of $\mathscr{L}_{\mathsf{F}}$ (with x as a free variable). Now let $\mathscr{L}_f$ be a standard first-order language comprising the same non-logical constants as $\mathscr{L}_{\mathsf{F}}$ as well as the function symbols $f_{\phi,\psi}$. We can define a translation of $\mathscr{L}_{\mathsf{F}}$ into $\mathscr{L}_f$ by recursively mapping each formula $\mathsf{F}\, x(\phi, \psi)$ into the assertion that $f_{\phi,\psi}$ is an injection of ϕ into ψ. Finally, let the theory T_f assert (i) the existence of identity maps $f_{\phi,\phi}$ for every ϕ; and (ii) the fact that if $f_{\phi,\psi}$ injects the ϕ's into the ψ's and y is not a ϕ and z is not a ψ then $f_{\phi\vee x=y,\psi\vee x=z}$ injects $\phi \vee x = y$ into $\psi \vee x = z$. Then, if θ is valid in all general models for $\mathscr{L}_{\mathsf{F}}$, its translation θ^f will be a consequence of T_f.

Finally, we notice that given the closure condition on the class $\mathcal{F}$ of functions associated with each general model (more specifically the fact that

[16] First-order quantifiers under the general interpretation — including the Frege quantifier — will be the subject of a separate treatment, which will include completeness proofs of the sort mentioned above.

each model always contains the empty injection), the abbreviations for $\forall$ and $\exists$ do have their intended meaning. In other words it will not be the case that $\forall x\phi(x)$ is true when something falls under $\neg\phi$, and conversely.

7 Formalizing Arithmetic

We are now ready to present the formalization of arithmetic. We will rely on two devices: the Frege quantifier (on either the standard or general interpretation) and an abstraction operator Num of the Xy kind mapping formulas into objects: $\mathsf{Num}(\phi)$ picks out an object x which is construed as the number of ϕ.[17] Since Num is an operator of the Xy kind, the value x it assigns to a formula ϕ is a first-order object — a member of D. This is the only constraint impose on $\mathsf{Num}(\phi)$. In particular, there is no requirement that Num be homogeneous (nor it could be, since its arguments and values are of different types), or, *a fortiori*, internal. The ultimate nature of "numbers" is left completely unspecified in the present account, but emphasis is placed on their cardinal properties.

We now turn to the task of providing axioms for arithemtic, using the above-mentioned devices. One option, of course, since $\mathcal{L}_{\mathsf{F}}$ interprets first-order logic, is just to reproduce the Peano-Dedekind axioms. But equally obviously, nothing could be farther from the spirit of the current enterprise. Rather, we want to establish arithmetic firmly on the ground of the cardinal properties of numbers, allowing then the *structural* properties of the latter to emerge as a consequence of the former, through a judicious use of abstraction. Furthermore, in keeping with the Fregean intuition that cardinality is a logical notion, we want to exploit as much as we can the expressive power of the Frege quantifier.

The axioms we consider are formulated in the pure language of identity, and conveniently divided into two groups. The first groups comprises axioms that do not have significant existential import. These axioms either lay down the basic definitions or impose identiy conditions on the entities involved. The second group of axioms more directly characterize the arithmetical universe, by specifying which entities are required, either directly or as a result of closure conditions, on the universe itself.

[17] Antonelli and May [2005] use a similar device, but there abstraction was represented by means of a heterogeneous *predicate* $\mathsf{VR}(x, \phi)$ representing the fact that x is the value range of ϕ. Since the abstraction principles dealt directly with extensions, care was taken that not all ϕ's were assigned a value-range, lest a paradox would arise. In keeping with the *non*-logical character of abstraction, that paper contains a particular well-motivated choice for concepts having value ranges.

I. *Definitional and uniqueness axioms*:

- EGFE for Num ("Hume's Principle"):

$$\mathsf{I}z(\phi(z), \psi(z)) \leftrightarrow \mathsf{Num}(\phi) = \mathsf{Num}(\psi)$$

- The number of ψ's succeeds the number of ϕ's:

$$\mathsf{Succ}(\phi, \psi) \leftrightarrow \exists x(\psi(x) \wedge \mathsf{I}y(\phi(y), \psi(y) \wedge y \neq x);$$

- Definition of the "less-than" relation:[18]

$$\mathsf{Num}(\phi) \leq \mathsf{Num}(\psi) \leftrightarrow \mathsf{F}\, z(\phi(z), \psi(z))).$$

- Definition of "x is a natural number":

$$\forall x(\mathbb{N}(x) \leftrightarrow \mathsf{Fin}\, y(\mathbb{N}(y) \wedge y \leq x)) \wedge x = \mathsf{Num}(\mathbb{N}(y) \wedge y \leq x)$$

  The last axiom asserts that x is a natural numbers if and only if x is the number of the concept "natural number less than x" and moreover such a concept is finite.

II. *Existence and closure axioms*:

- The infinitary axiom:

$$\forall x(\phi(x) \rightarrow \exists! y(\psi(y) \wedge \theta(x, y))) \rightarrow \mathsf{F}\, x(\phi(x), \psi(x)).$$

  The axiom expresses the closure of the set $\mathcal{F}$ of injections under definability, and therefore subsumes the existence of the empty and identity maps.

- Principle of Counting:

$$\mathsf{Succ}(\phi, \mathbb{N}(y) \wedge y \leq \mathsf{Num}(\phi)).$$

  The definition of natural number tells us that each number x is the number assigned to the set of its predecessors; the principle of counting tells us that if n numbers the ϕ's, then the number of the concept "predecessor of n" is $n + 1$.

[18] If we further assume that $\leq$ is anti-symmetric then this definition of $\leq$ implies HP. Alternatively, the anti-symmetry of $\leq$ follows from the definition together with HP.

- Principle of Induction, in the form "Every finite, non empty set of numbers has a maximum":

$$[\exists x(\mathbb{N}(x) \wedge \phi(x)) \wedge \mathsf{Fin}\, x(\mathbb{N}(x) \wedge \phi(x))] \rightarrow \\ \exists y[(\mathbb{N}(y) \wedge \phi(y)) \wedge \forall x(\mathbb{N}(x) \wedge \phi(x) \rightarrow x \leq y)].$$

These axioms allow the representation of a great many facts about arithmetic. In fact, on the standard interpreation of F, they are categorical and completely characterize the standard model of arithmetic. But also on the general interpretation, when the language is appropriately augmented with $+$ and $\times$, the theory interprets Peano arithmetic. As an example of this, critically using the infinitay axiom, we proceed to prove the induction schema. The schema has the form:

$$\forall n((\forall m < n)\phi(m) \rightarrow \phi(n)) \rightarrow \forall n\phi(n),$$

where quantifiers such as $\forall n(\ldots)$ abbreviate $\forall x(\mathbb{N}(x) \rightarrow \ldots)$. First let 0 denote the unique x such that $x = \mathsf{Num}(y \neq y)$, and then notice that, since injections in $\mathcal{F}$ are assumed to satisfy the infinitary axiom, the following is provable:

$$(*) \qquad \mathsf{Fin}\, x\ \phi(x) \wedge \forall x(\psi(x) \rightarrow \phi(x)) \rightarrow \mathsf{Fin}\, x\ \psi(x).$$

Now assume the antecedent of the induction principle, i.e.,

$$\forall n((\forall m < n)\phi(m) \rightarrow \phi(n)),$$

but suppose the consequence fails: $\neg\forall n\phi(n)$. Choose n such that $\neg\phi(n)$. In particular, $\mathsf{Fin}\, y(\mathbb{N}(y) \wedge y \leq n)$.

Now let $S = \{m < n : (\forall z < m)\phi(z)\}$. Since $0 \in S$, it follows that S is non-empty. Further, everything in S satisfies $\mathbb{N}(x) \wedge \phi(x)$, whence also $\mathsf{Fin}\, x(x \in S)$ by $(*)$. By the induction axiom, let $p = \max S$. Then (since $p \in S$), $p < n$, i.e., $p + 1 \leq n$, and moreover $p + 1 \notin S$. From $p + 1 \leq n$ we distinguish two subcases:

Case 1 $p + 1 < n$; then $p + 1 \in \{M < n : (\forall z < m)\phi(z)\} = S$, which is impossible.

Case 2 $p + 1 = n$; then again $(\forall z < p + 1)\phi(z)$, which by the assumption gives $\phi(p + 1)$, i.e., $\phi(n)$, also impossible.

Other properties of the natural numbers are derivable in a similar fashion.

The account just developed heavily relies on cardinality notions, exploiting the fact that such notions can — to an extent at least — be treated at the first order. It is worth noting that there does not seem to be any obvious way to extend the present treatment to *ordinal* notions, except trivially in finite domains, where ordinal and cardinal numbers coincide. In this respect ordinal notions, while ordinarily regarded on a par with their cardinal counterparts, appear instead to be intrinsically more complex than the latter, and indeed quite possibly beyond the reach of a first-order treatment.

8 Conclusion

After identifying abstraction as a particular EGFE principle and formulating logicism as a most general claim about the logical character of cardinality notions, we have developed an account of arithmetic that satisfies all three *desiderata* set out at the beginning of the paper:

1. The account is driven by the cardinal properties of the natural numbers, and derives the structural properties from the latter, rather than the other way around as is the case, for instance, with set-theoretic reductions.

2. The account does justice to the Frege-Russell characterization of the natural numbers as either identical, or in any case intimately connected, to equinumerosity classes.

3. The account proceeds entirely at the first order from a semantical point of view, a fact that is even more evident if we adopt the general interpretation of the Frege quantifier.

But what *are* numbers, on the present account? We have indeed left the "ultimate nature" of numbers (whatever that might mean) completely unspecified, as already noted. Does that mean that Benacerraf's [1965] conclusion that "any ω-sequence will do after all" is the correct way to approach a theory of arithmetic?.

A word of caution is in order here. In some sense, Benacerraf's answer is unsatisfying: being an ω-sequence is a structural property *par excellence*, and such properties are best viewed as supervenient upon cardinal ones. So Benacerraf'a account is incomplete. But the account cannot be completed by a characterization of the ontological status of numbers, a determination of which entities, among the many that populate our mathematical universe, are to regarded as the numbers that in Kronecker's words were divinely given to us.[19] But there is a sense in which Benacerraf's point is completely right: once we have an account of natural numbers in terms of cardinal properties, it does not make any difference which objects are chosen as representatives of the equinumerosity classes, as long as we have enough of them to satisfy the inflationary thrust of the corresponding abstraction principle. But no matter what objects are chosen, they will form an ω-sequence, so that, in this sense, any ω-sequence will do after all. These considerations also lead to a deflation of general worries (nominalistic or otherwise) concerning abstract objects in general, and numbers in particular. On the present account, there is no separate realm of abstract objects, and we should not have any qualms appelaing to them for whatever philosophical, logical, or mathematical purpose we might be pursuing: abstract objects are just ordinary objects that have been recruited for the purpose of serving as representatives of certain equivalence classes.

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