Frege's *other* program

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Abstract

Frege's logicist program requires that arithmetic be reduced to logic. Such a program has recently been revamped by the "neo-logicist" approach of Hale & Wright. Less attention has been given to Frege's extensionalist program, according to which arithmetic is to be reconstructed in terms of a theory of extensions of concepts. This paper deals just with such a theory. We present a system of second-order logic augmented with a predicate representing the fact that an object x is the extension of a concept C, together with extra-logical axioms governing such a predicate, and show that arithmetic can be obtained in such a framework. As a philosophical payoff, we investigate the status of so-called "Hume's Principle," and its connections to the root of the contradiction in Frege's system.

1 Introduction

Two distinct research strands come together in Frege's program, as articulated in *Grundlagen* and, in finer detail, in *Grundgesetze.* The first is the idea that arithmetic should be reducible (in a suitable sense) to *logic*; the second is the idea that arithmetic should be recoverable from a theory of *extensions.* Frege's work lies at the intersection of these two programs, and it amounts to the idea that arithmetic should be reconstructed in terms of a *logical* theory of *extensions.* The first program is traditionally referred to as *logicism*, while the second could be referred to — for wont of a better term — as *extensionalism.*

As we now know, if consistency is to be preserved, logicism and extensionalism as conceived by Frege cannot be pursued simultaneously. Something has to give, and there are a number of options available. One alternative explored by Heck [1996]; Wehmeier [1999]; Ferreira and Wehmeier [2002] maintains the coordination of logicism and extensionalism. Consistency is re-established by weakening the comprehension principle, so that the offending predicates, e.g., the Russell predicate, no longer fall under the principle. In effect, in the presence of a schematic version of Frege's Basic Law V such as Heck and Wehmeier have, this strategy ultimately limits the number of predicates having value ranges. The problem with this approach is that, as far as is currently known, it is too weak as it does not entail Peano Arithmetic, but only weaker systems, in particular Robinson's Q.

Alternatively, unlike the approach just described, we can break the bond of logicism and extensionalism, rejecting one while maintaining the other. The most well-known approach that arguably maintains logicism is the *neo-logicist* program of Hale and Wright [2001]. They abandon the idea that arithmetic can be obtained from a theory of extensions; they introduce in its stead a theory of numbers based on an operator mapping each concept F to an object x, construed as "the number of F." In order for this mapping to be interpreted as an assignment of numbers to concepts, certain further constraints must be satisfied: not just any mapping of the concepts into the objects will do. In particular, it is assumed that such a mapping satisfies what has come to be known (perhaps improperly) as *Hume's Principle*: concepts F and G are mapped to the same object precisely when there is a one-to-one correspondence between the objects that fall under F and the objects that fall under G. In *Grundgesetze*, Frege shows that Hume's Principle, initially introduced in *Grundlagen*, is entailed by Basic Law V. Wright and Hale, however, reject Basic Law V, and take Hume's Principle as basic. They then aver to the result that Peano Arithmetic can be derived in second-order logic with the addition of Hume's Principle (the so-called "Frege's Theorem"[1]). Since Hale and Wright eschew Basic Law V, the neo-logicist system is consistent (more precisely, equiconsistent with second-order arithmetic). The burden of proof that the resulting system is really *logicism* falls squarely on its proponents showing that the mapping guaranteed by Hume's principle is purely a matter of logic (or at least that it is constitutive of the notion of number), although the resolution of this issue, for reasons that will become apparent, will not detain us here.[2]

One consequence of neo-logicism is that numbers are (logical) objects, a position also compatible with the approaches of Heck and Wehmeier. There is an alternative way to understand numbers from within a Fregean point of view, however, which is that numbers are not objects, but concepts. To take this point of view, however, one must have a theory of extensions, that is, we need to embrace the *other* half of Frege's program — extensionalism.

[1]See, for instance, Heck [1999] for an informal overview.

[2]We will not address here the question of the extent to which this part of the neo-logicist program is successful (but the interested reader can consult Wright [1999] and, for an opposing viewpoint, Boolos [1997]).

This is what we aim to do in this paper, to show in a *non-logical* theory of extensions, where numbers are concepts, not objects, that Peano Arithmetic can be derived. We will be exploring the feasibility of the extensionalist program as a genuine theory of concept extensions, within the framework of a second-order theory that deals explicitly with concepts. Within such a theory, an object x can be regarded as the extension (or — as we will also say — the *value range)* of a concept F provided it satisfies a constraint analogous to Frege's Basic Law V: *if* concepts F and G have extensions, then their extensions are the same precisely when the same objects fall under F as fall under G. It will thus be a notable characteristic of our theory that it is a consistent theory which incorporates both a version of Basic Law V and an unrestricted Comprehension Principle.[3]

The map we have just given of the possible responses to the contradiction in Frege's system is exhaustive. However, each alternative places the roots of the inconsistency in rather different lights. We know, after the paradoxes, that not every predicate can have a value range: this is the import of Cantor's theorem. At the same time, we know that at least some predicates must have value ranges, if arithmetic is to be derived. Hence there is a tension between the existential assumptions needed for arithmetic and the limitations imposed by Cantor's theorem. What we would want an analysis to provide is real insight into how these existential requirements needed for a development of arithmetic play along with the constraints imposed by Cantor's result.

In regard to this matter, the neo-logicist approach falls short: it attains consistency, if you will, by brute force, by expunging Basic Law V, but this hardly provides any real insight. All it gives us is a consistent theory from which arithmetic can be derived. Of course, this is a desirable result, but what one would like is a more subtle account of the way in which the map taking predicates into the objects needs to be constrained, while still satisfying the Peano-Dedekind axioms.

There are two ways in which one can strike a balance between the existential requirements and consistency. One is by limiting Comprehension, i.e., by denying that every predicate corresponds to a concept. This is the approach advocated by Heck and Wehmeier, who limit Comprehension to (little more than) its predicative instances.[4] The alternative is to preserve Comprehension in its full generality, but to make explicit existential assumptions as to what concepts have value ranges. This is the approach that we will develop.

Of the two approaches, one maintains the purely logical character of the system, but gives

[3]In this paper, we take the Comprehension Principle to express the fact that each predicate determines a concept. For present purposes, a *predicate* can be interpreted as whatever corresponds (semantically) to an open formula.

[4]Whereas Heck's system is strictly predicative, allowing only instances of Comprehension with no bound second-order variables, Wehmeier can push the result to the Δ^1_1 fragment of Comprehension, thereby achieving an optimal result.

up on the generality of Comprehension; the other maintains Comprehension but assumes a partial map of the concepts into the objects governed by extra-logical principles. But there are important differences between these approaches. First and foremost, as we shall see, only in the latter system is Peano Arithmetic derivable; while as noted, in the former, only Robinson's Q is known to be derivable. Second, the latter system's explicit existential assumptions tell us something about the mathematical universe and the objects that inhabit it, as opposed to limiting Comprehension, which just constrains the conceptual space.

A third difference, one that will differentiate our approach both from the proposal currently under discussion and neo-logicism, is that Hume's Principle can be shown not to follow. We show this by exhibiting a counter-example to the left-to-right direction, i.e., we show that there can be concepts with the same number that are not equinumerous. Such a possibility is precluded if numbers are conceived of as objects, and in fact our counter-example crucially depends on assumptions as to what concepts have value ranges. The importance of this sort of counter-example is that it goes straight to the heart of the matter as regards the origin of the contradiction, in ways that other counter-examples to Hume's Principle do not (as we will see).

The paper is articulated as follows. In the next section, we will present an informal overview of the argument; in the section to follow we will provide the formal details. We will then draw out the implications of the approach for the issues we have been elucidating with particular attention to the status of Hume's Principle.

2 Extensionality: the informal development

As we set out to outline our strategy, one question naturally arises: isn't modern set theory already a stunningly successful articulation of the extensionalist program? Insofar as (presumably) consistent fragments of *Grundgesetze* go, Zermelo-Fraenkel set theory goes, undeniably, a long way. But, we contend, modern set theory is extensionalism *in disguise*: although it is a theory of extensions of concepts, gone are the concepts, and only the extensions are left.

On the other hand, we want to pursue a genuine extensionalist program in which concepts, along with their extensions, are first-class citizens. (Many of the ideas at work here were developed in connection with a reconstruction of Frege's own approach to logical meta-theory, his so-called "New Science" — see Antonelli and May [2000].) This requires a second-order language that allows explicit quantification over predicates, relations, etc., as well ordinary first-order quantification. Additionally, we will need to assume that the language contains a predicate constant VR that relates concept variables to individual ones, with the understanding that $\mathsf{VR}(P, x)$ expresses that the object x is the value range of the concept

P. We will *not* assume that every concept has a value range; rather, which concepts do is a matter of independent stipulation. But before we explain what those stipulations are, we need to go over the analysis of the concept of number.

In the framework we just outlined, arithmetical notions such as those of *number* and *natural number* can be defined. However, some design choices need to be made, and in particular an issue arises as to the status of numbers, *viz.*, whether they should be identified with certain objects or with other higher-level entities. On Frege's general approach, numbers are identified with equivalence classes of equinumerous concepts. As such, numbers are third-level concepts.[5] It is well known, however, that Frege employs the injection of the concepts into the objects given by Basic Law V in order to reduce the level of the numbers by identifying them with the *extensions* of the third-level concepts.

The question, then, is whether this is a mere convenience or whether this reduction plays an essential role in Frege's program. As will become clear, our view is that although some injection of the concepts into the objects is necessary to get arithmetic going, this does not necessarily take the form of an indentification of the *numbers* themselves with certain particular objects or value ranges, and in particular the objects that provide the target of the map of the concepts into the objects need not be construed as the numbers themselves. In fact, Frege himself repeatedly says that very many uses of Basic Law V are dispensable; among them, we submit, is the one that leads to the identification of the numbers as objects.

On our account, then, numbers are *concepts*, not *objects*.[6] Like Frege, we also make use of (a version of) Basic Law V to reduce the level of the hierarchy at which numbers can be found. But instead of performing a complete reduction from the third level to the first, we identify numbers with second-level concepts. As such, numbers are concepts under which first-level objects fall, but to retain the original Fregean account of numbers as equivalence classes of equinumerous concepts, we constrain the objects that are allowed to fall under the numbers to be value-ranges of equinumerous concepts. In other words, N is a number if and only if there is a concept P such that an object x falls under N if and only if x is the value-range of a concept equinumerous to P. Such an object x is called a *witness* for N.

With this characterization of numbers basic principles such as the schema of induction can then be proved. But more work needs to be done if Peano Arithmetic is to be recovered.

[5]Our usage throughout will be that objects are of the first level, n-ary concepts of objects are of the second level, n-ary concepts that take second-level concepts as arguments are of the third-level, and so on. We diverge from Frege, for whom only concepts and relations have an order. Thus our third-level concepts are Frege's second-level concepts.

[6]That is, numbers are concepts of the second level. This contrasts with the proposal of Hodes [1984] who proposes that numbers are concepts of one level higher; to wit, cardinality quantifiers that take second level concepts as arguments. While Hodes's goals are primarily philosophical, Rayo [2002] gives a formalization incorporating this view of numbers along with a proof of a "completeness theorem for applied arithmetic."

In particular, after laying down the language and its semantics, we can formulate the three characteristic axioms of the system. These axioms are essentially *non-logical* in nature (although, arguably, to different degrees). They are far from true in every model and are what provides the theory with it expressive power. The axioms are also conceptually quite simple.

The first axiom, a version of Basic Law V, establishes identity conditions for value ranges. It is formulated in such a way as not to have existential import: it says that *if* concepts P and Q have value ranges, then the value ranges are identical precisely when P and Q are equi-extensional. Nothing follows from this axiom as to the existence of value ranges. The second axiom is a second-order comprehension principle. It does have existential import, but only at the second level: it implies that, for any predicate expressed by a possibly complex formula Φ, there corresponds a *concept* P. Comprehension is a closure condition on the collection of the subsets of the domain. It has existential import, but not of the kind that is at work in any derivation of arithmetic. Finally, the third axiom is the one that expresses the theory's characteristic existential assumptions *at the first level*. These assumptions require that every concept under which only witnesses fall have a value range.

The system is sufficient to derive the Peano-Dedekind axioms. Much of the derivation at this point is rather straightfoward, with the exception of the claim that every number is distinct from its predecessors (theorem 3.19). It is worth noting that, given our definition of numbers as concepts, it can be proven on the basis of Comprehension alone that every number has a successor; what does not follow from comprehension, and requires the characteristic existential assumptions of the system, is the claim that every number is distinct from its successor. This is, then, a version of "Frege's theorem" as embodied in our theorem 3.19 below.

3 Extensionality: formal details

We can now proceed with the formal details. Our first step is to introduce a formal system $\mathcal{F}$, which will be used in articulating the extensionalist approach. The system $\mathcal{F}$ is a second-order system with a standard second-order comprehension axiom, as well as a characteristic extra-logical axiom schema ensuring the existence of certain value ranges.

The language $\mathcal{L}$ of $\mathcal{F}$ is a standard second-order language, comprising first- and second-order variables, predicate constants (among which at least $=$), connectives $\rightarrow$ and $\perp$ (the Falsum), and first- and second-order universal quantifiers $\forall x$ and $\forall P$ (the other connectives and quantifiers are to be regarded as abbreviations in the usual way). Ordinarily, at this point one would have an unrestricted abstraction operator assigning to each formula $\phi(x)$ a first-order term $\hat{x}.\phi(x)$. But the map assigning value ranges to concepts must, as we have seen, be *partial*. Rather than changing the underlying logic allowing for partial maps, we introduce a *relational* version of the extensional apparatus, allowing for the possibility that concepts might or might not have a value range. So we introduce a special predicate $\mathsf{VR}(P, x)$ with the intended interpretation that x is the value range of P. Whether a given concept does indeed have a value range needs then to be specified through specific assumptions, which we regard as *extra-logical* in nature.

Next, we supply an interpretation for $\mathcal{L}$. When interpreting a second-order language one is faced with the option of giving the *standard* interpretation (in which the n-ary second-order variables are taken to range over the true power set of D^n) or the *general* interpretation (in which the n-ary second-order variables are taken to range over some given collection of subsets of D^n). As nothing we are going to say depends on the interpretation being standard, we opt for the weaker general alternative.

Accordingly, we define a model $\mathfrak{M}$ for $\mathcal{L}$ to comprise a non-empty first-order domain D as well as, for each n, a non-empty collection of subsets of D^n (providing a range for the n-place second-order variables). In such a model, each n-place predicate constant (if there are any), is assigned a particular subset of D^n, and the characteristic non-logical constant of $\mathcal{L}$, VR, is interpreted by a function assigning a subset of D of cardinality ≤ 1 to each predicate variable P. (The restriction on cardinality correspond to the intuition that concepts cannot be assigned more than one value range.)

In practice, we will be interested in models that satisfy certain further closure conditions, such as are needed, for instance, for the second-order domains to satisfy comprehension. Before we can specify the axioms that will constrain the class of models, though, we need to develop some arithmetical notions.

3.1 Notation We have the following conventions:

- If P and Q are n-place predicate variables and$\overline{x} = x_1, \ldots, x_n$, we abbreviate $\forall \overline{x}(P\overline{x} \leftrightarrow Q\overline{x})$ by $P = Q$; $\forall \overline{x}(P\overline{x} \rightarrow Q\overline{x})$ by $P \preceq Q$; and $P \preceq Q \wedge P \neq Q$ by $P \prec Q$. Observe that the semantics validates the inference from $\Phi(P)$ and $P = Q$ to $\Phi(Q)$, for any formula Φ.
- If $\phi(\overline{x})$ is a formula, we write $\phi = P$ to abbreviate $\forall \overline{x}(P\overline{x} \leftrightarrow \phi(\overline{x}))$.
- We abbreviate $\exists P[\forall \overline{y}(P\overline{y} \leftrightarrow \phi(\overline{y})) \wedge \mathsf{VR}(P, x)]$ by $\mathsf{VR}(\phi, x)$.

3.2 Definition Let $P \approx Q$ abbreviate the standard claim that there is a 1-1 correspondence between the P's and the Q's:

$$\begin{aligned}\exists R[\ & \forall x(Px \rightarrow \exists y(Qy \wedge Rxy) \wedge \forall y(Qy \rightarrow \exists x(Px \wedge Rxy) \wedge \\ & \forall x(Px \rightarrow \forall u \forall v(Qu \wedge Qv \wedge Rxu \wedge Rxv \rightarrow u = v) \wedge \\ & \forall y(Qy \rightarrow \forall u \forall v(Pu \wedge Pv \wedge Ruy \wedge Rvy \rightarrow u = v) \quad]\end{aligned}$$

3.3 DEFINITION Define $\mathsf{N}(P,Q)$, "Q is the number of P's," if and only if Q is the concept 'y is the value-range of a concept $S \approx P$':

$$\forall y(Qy \leftrightarrow \exists S(\mathsf{VR}(S,y) \wedge P \approx S)).$$

We use second-order variables $N, M, P, \ldots$ for numbers, and by a slight abuse of notation, we also write $\mathsf{N}(N)$, "N is a number" as $\exists P\, \mathsf{N}(P,N)$. We also write $\mathsf{Z}(N)$, "N is zero," if and only if $\exists P(\forall y \neg P(y) \wedge \mathsf{N}(P,N))$, i.e., N is the number of an empty concept.

After we introduce the characteristic axioms of the system, we will be in a position to establish the elementary properties of the natural numbers, such as existence and uniqueness of zero:

$$\exists N[\mathsf{Z}(N) \wedge \forall M(\mathsf{Z}(M) = N = M)],$$

and many more. For now we observe that the result will allow us to introduce a predicate constant for the number zero, also denoted by Z. Next, we define the relation of (immediate) successor for numbers. The definition follows Frege's original definition according to which N is the immediate successor of M if and only if M is the number of a concept P under which an object x falls, and N is the number of the concept "falling under P but other than x."

3.4 DEFINITION Define $\mathsf{Sc}(M,N)$, "N is the (immediate) successor of M," as follows:

$$\exists P \exists Q \exists z[\mathsf{N}(P,M) \wedge \mathsf{N}(Q,N) \wedge Qz \wedge \forall w(Pw \leftrightarrow Qw \wedge w \neq z)].$$

Our next task is to define the notion of a *natural number*. A particular important notion throughout will be the idea that an object x is a *witness* for a number N: this will happen if x is the value range of a concept having that number. Formally:

3.5 DEFINITION $\mathsf{Wtn}(N,x)$ if and only if $\exists P(\mathsf{N}(P,N) \wedge \mathsf{VR}(P,x)$. Notice that this definition is equivalent to $\mathsf{N}(N) \wedge N(x)$.

Our development of arithmetic (on the basis of the axioms given in this section) will rely — crucially — on the fact that every natural number will turn out to have a witness, and this in turn will require that we carefully select special witnesses for the natural numbers.

Our definition of *natural number* follows the standard (higher-order) inductive definition: N is a natural number if and only if every concept S which contains a witness for zero and such that if it contains a witness for M then it contains witnesses for any successors of M, it also contains a witness for N. (We use the plural "successors" in the above paraphrase because we have not proved yet that successors are unique.)

3.6 Definition We define $\mathsf{Nn}(N)$, "N is a natural number," as follows:

$$\begin{aligned}\forall S[\exists y(\mathsf{Wtn}(\mathsf{Z},y)\wedge Sy)\wedge\forall M(\mathsf{N}(M)\wedge\exists y(\mathsf{Wtn}(M,y)\wedge Sy)\\ \rightarrow\forall M'(\mathsf{Sc}(M,M')\rightarrow\exists y(\mathsf{Wtn}(M',y)\wedge Sy)))\\ \rightarrow\exists y(\mathsf{Wtn}(N,y)\wedge Sy)]\end{aligned}$$

Notice the occurence of the constant Z in the above definition. Although the introduction of Z has not been justified yet, it's easy to see how to reformulate the definition in such a way that Z does not occur in it.

This definition will allow us to prove a principle of induction on Nn. It will be expedient — for the sake of readability — to introduce an abbreviation for the recurring formula $\exists y(\mathsf{Wtn}(N,y)\wedge Sy)$; so let us use $\mathsf{WTN}(N,S)$ to mean that S contains a witness for N. With this abbreviation, the definition of Nn becomes:

$$\begin{aligned}\forall S\Big[[\mathsf{WTN}(\mathsf{Z},S)\wedge\\ \forall M(\mathsf{N}(M)\wedge\mathsf{WTN}(M,S)\rightarrow\forall M'(\mathsf{Sc}(M,M')\rightarrow\mathsf{WTN}(M',S)))]\\ \rightarrow\mathsf{WTN}(N,S)\Big]\end{aligned}$$

3.7 Theorem The following induction principle is valid:

$$\begin{aligned}\forall S\Big[[\mathsf{WTN}(\mathsf{Z},S)\wedge\\ \forall M(\mathsf{N}(M)\wedge\mathsf{WTN}(M,S)\rightarrow\forall M'(\mathsf{Sc}(M,M')\rightarrow\mathsf{WTN}(M',S)))]\\ \rightarrow\forall N(\mathsf{Nn}(N)\rightarrow\mathsf{WTN}(N,S))\Big]\end{aligned}$$

The above induction principle follows analytically from the definitions. This does not tell us anything about what natural numbers there are, a question that will in turn depend on which concepts have value ranges.

We now present the three characteristic axioms of the theory $\mathcal{F}$.

F1 A version of Frege's "Basic Law V" (BLV):

$$\forall P\forall Q\forall x\forall y[\mathsf{VR}(P,x)\wedge\mathsf{VR}(Q,y)\rightarrow(\forall\overline{z}(P\overline{z}\leftrightarrow Q\overline{z})\leftrightarrow x=y)];$$

F2 A comprehension principle for any formula ϕ (with occurrences of the constant VR and free parameters other than P allowed in ϕ): $\exists P\forall\overline{x}[P\overline{x}\leftrightarrow\phi(\overline{x})]$;

F3 Special existential axioms providing for the existence of value ranges: for any formula ϕ,

$$\forall x(\phi(x)\rightarrow\exists M(\mathsf{Nn}(M)\wedge\mathsf{Wtn}(M,x)))\rightarrow\exists x\,\mathsf{VR}(\phi,x).$$

This last axiom is the crucial one: it guarantees that any (possibly complex) predicates that only apply to witnesses of natural numbers have value ranges. Also, notice that **F1** implies that any concept has at most one VR: $\forall P \forall x \forall y[\mathsf{VR}(P,x) \wedge \mathsf{VR}(P,y) \rightarrow x = y]$.

As a first example of how these axioms can be used in the derivation of arithmetical principles, we make good on our promise on the uniqueness of the number zero.

3.8 THEOREM $\exists N(\mathsf{Z}(N) \wedge \forall M(\mathsf{Z}(M) \leftrightarrow N = M))$.

Proof. By Comprehension, there is a P such that $\forall x \neg Px$; then (vacuously) $Px \rightarrow \exists M(\mathsf{Nn}(M) \wedge \mathsf{Wtn}(M,x))$. Then, by the special axiom, P has a value range, denoted (here and henceforth) $\mathbf{z}$. Moreover, $\mathbf{z}$ is also the value range of any other empty predicate: if y is such that

$$\exists Q[\forall x \neg Qx \wedge \mathsf{VR}(Q,y)],$$

then $y = \mathbf{z}$ by BLV. By comprehension again let Nx hold if and only if $x = \mathbf{z}$; then $\mathsf{N}(P,N)$. This shows $\exists N\, \mathsf{Z}(N)$. For uniqueness:

$$\begin{aligned}
\mathsf{Z}(M) &\leftrightarrow \exists P[\forall x \neg Px \wedge \mathsf{N}(M,P)] \\
&\leftrightarrow \exists P[\forall x \neg Px \wedge \forall y(My \leftrightarrow \exists S(S \approx P \wedge \mathsf{VR}(S,y)))] \\
&\leftrightarrow \forall y(My \leftrightarrow \exists S(\forall x \neg Sx \wedge \mathsf{VR}(S,y)))] \\
&\leftrightarrow \forall y[My \leftrightarrow y = \mathbf{z}] \\
&\leftrightarrow M = N.
\end{aligned}$$

■

In view of the above proof, from now on we introduce the predicate constant Z to denote the unique Q such that $\mathsf{Z}(Q)$.

Similarly, we can now define a relation $\leq$ between numbers, as usual, by appeal to the ancestral of Sc.

3.9 DEFINITION $M \leq N$ holds if and only if every set which contains a witness for M and is closed under witnesses of successors, contains a witness for N:

$$\begin{aligned}
&\forall S[\exists y(\mathsf{Wtn}(M,y) \wedge Sy) \wedge \forall M'(\exists y(\mathsf{Wtn}(M',y) \wedge Sy) \\
&\qquad \rightarrow \forall M''(\mathsf{Sc}(M',M'') \rightarrow \exists y(\mathsf{Wtn}(M'',y) \wedge Sy))) \\
&\quad \rightarrow \exists y(\mathsf{Wtn}(N,y) \wedge Sy)]
\end{aligned}$$

3.10 THEOREM If $M \leq N$ and $N \leq M$ then $M = N$.

Proof. Double induction on M, N. ■

We now introduce the Peano-Dedekind axioms for first-order arithmetic, and show that they can be derived on the basis of the proposed framework. The proof will also make clear exactly how the characteristics axiom *F3* is to be used. Especially the argument for theorem 3.19 is particularly representative of how the kind of recursive "feedback loop" that the assumption on witnesses engenders.

Consider the following axiomatization of arithmetic (analogous, for instance, to the one in Boolos and Heck [1998]) :

1. $\exists N(\mathsf{Z}(N) \wedge \mathsf{Nn}(N))$;
2. $\forall N \forall M(\mathsf{Nn}(N) \wedge \mathsf{Sc}(N, M) \rightarrow \mathsf{Nn}(M))$;
3. $\forall M \forall N_1 \forall N_2(\mathsf{Nn}(M) \wedge \mathsf{Sc}(M, N_1) \wedge \mathsf{Sc}(M, N_2) \rightarrow N_1 = N_2)$;
4. $\forall N(\mathsf{Nn}(N) \rightarrow \exists M(\mathsf{Sc}(N, M) \wedge N \neq M))$;
5. $\forall N \forall M(\mathsf{Nn}(N) \wedge \mathsf{Z}(M) \rightarrow \neg \mathsf{Sc}(N, M))$;
6. $\forall M \forall N_1 \forall N_2(\mathsf{Nn}(N_1) \wedge \mathsf{Nn}(N_2) \wedge \mathsf{Sc}(N_1, M) \wedge \mathsf{Sc}(N_2, M) \rightarrow N_1 = N_2)$;
7. For every formula $\Phi(X)$ with the free second-order variable X:

$$\Phi(\mathsf{Z}) \wedge \forall N(\mathsf{Nn}(N) \wedge \Phi(N) \rightarrow \forall M(\mathsf{Sc}(N, M) \rightarrow \Phi(M))) \rightarrow \forall N(\mathsf{Nn}(N) \rightarrow \Phi(N)).$$

Notice that this last is an axiom schema. Equivalently (in the presence of the comprehension and the special existential axioms) we could have the single axiom that every P containing a witness for Z and closed under witnesses of successors, contains a witness for every natural number. We begin our verification that the axioms hold from this last one.

3.11 Theorem The following induction principle holds, for any formula $\Phi(P)$:

$$\Phi(\mathsf{Z}) \wedge \forall N(\mathsf{Nn}(N) \wedge \Phi(N) \rightarrow \forall M(\mathsf{Sc}(N, M) \rightarrow \Phi(M))) \rightarrow \forall N(\mathsf{Nn}(N) \rightarrow \Phi(N)).$$

Proof. Recall $\mathsf{Nn}(N)$ holds if every predicate S which contains a witness for Z and such that if it contains a witness for M then it contains a witness for any successors of M, contains a witness for N. To show the validity of the above principle, use comprehension (**F3**) to obtain a predicate S such that

$$\forall x[Sx \leftrightarrow \exists M(\mathsf{Nn}(M) \wedge \mathsf{Wtn}(M, x) \wedge \Phi(M))],$$

and apply the form of induction of theorem 3.7. ■

Next, we take up all the remaining axioms, leaving for last the crucial fourth axiom, asserting that every number has a successor (other than itself).

3.12 Theorem $\exists N(\mathsf{Z}(N) \wedge \mathsf{Nn}(N))$.

Proof. We already know from theorem 3.8, that there exists a unique N which is the number of the empty predicate. Obviously, this N is a natural number since its witness $\mathbf{z}$ belongs to every S which contains a witness for N and is closed under witnesses of successors. ■

3.13 Theorem The following arithmetical axioms hold in $\mathcal{F}$:

1. $\forall N \forall M(\mathsf{Nn}(N) \wedge \mathsf{Sc}(N, M) \rightarrow \mathsf{Nn}(M))$;
2. $\forall M \forall N_1 \forall N_2(\mathsf{Nn}(M) \wedge \mathsf{Sc}(M, N_1) \wedge \mathsf{Sc}(M, N_2) \rightarrow M = N)$
3. $\forall N \forall M(\mathsf{Nn}(N) \wedge \mathsf{Z}(M) \rightarrow \neg \mathsf{Sc}(N, M))$
4. $\forall M \forall N_1 \forall N_2(\mathsf{Nn}(N_1) \wedge \mathsf{Nn}(N_2) \wedge \mathsf{Sc}(N_1, M) \wedge \mathsf{Sc}(N_2, M) \rightarrow N_1 = N_2)$

Proof. To be supplied, but see Boolos and Heck [1998] for details of similar proofs. ■

As long as the first-order domain of the model is actually infinite, it will be possible to prove that every number has a successor; however, if there not enough value ranges around, it won't necessarily follow that $\mathsf{Sc}(N, M)$ implies $N \neq M$.

In the development of arithmetic the crucial axiom is axiom 4, which guarantees that every number has a successor (other than itself). Recall that we introduced $\mathbf{z}$ as the value range of the empty predicate; we also have:

3.14 Lemma $\mathsf{Wtn}(\mathsf{Z}, \mathbf{z})$.

Our next task is to select, for each number N, a special witness for N. The series of witnesses will very much resemble the von Neumann (finite) ordinals, in that each one will be the value range of the concept that applies to all previous witnesses.

3.15 Definition Let $\mathbf{N}$ be the smallest predicate P containing $\mathbf{z}$ and satisfying the closure condition:

$$\forall Q \forall x \forall S \forall y[Px \wedge \mathsf{VR}(Q, x) \wedge \forall z(Sz \leftrightarrow (Qz \vee z = x)) \wedge \mathsf{VR}(S, y) \rightarrow Py].$$

Here is an intuitive picture of what $\mathbf{N}$ looks like. The concept $\mathbf{N}$ will be a sequence of witnesses $n_0, n_1, n_2, \ldots$, where n_k is the value range of the predicate "$x = n_i$, for some $i < k$."

More precisely (where VR is now — by abuse of language — an operator, λ-notation is used to denote predicates):

$$\begin{aligned} n_0 &= \mathsf{VR}(\lambda y \mathbin{.} y \neq y) = \mathbf{z} \\ n_1 &= \mathsf{VR}(\lambda y \mathbin{.} y \neq y \vee y = n_0) = \mathsf{VR}(\lambda y.y = n_0) \\ n_2 &= \mathsf{VR}(\lambda y \mathbin{.} y = n_0 \vee y = n_1) \\ &\vdots \\ n_{k+1} &= \mathsf{VR}(\lambda y \mathbin{.} y = n_0 \vee \ldots \vee y = n_k) \end{aligned}$$

3.16 Lemma $\forall x(\mathbf{N}x \rightarrow \exists N(\mathsf{Nn}(N) \wedge \mathsf{Wtn}(N,x)))$

Proof. By induction on the generation of $\mathbf{N}$ (which is provable). As observed, $\mathbf{z}$ is a witness for Z. If x is in $\mathbf{N}$, where $\mathsf{VR}(Q,x)$, and Q is a predicate of witnesses having number N, then the predicate $\lambda y \mathbin{.} Qy \vee y = x$ has number $\leq N+1$ (where the inequality could be strict if already Qx — although we will prove that this is not the case; the idea here is if $n_0, \ldots, n_k$ are witnesses, then $\lambda y \mathbin{.} y = n_0 \vee \ldots \vee y = n_k$ has number $\leq k+1$). But then $\lambda y \mathbin{.} Qy \vee y = x$ also applies only witnesses, and hence by the special existential axiom has a value range y in $\mathbf{N}$, witnessing a number $\leq N+1$. ∎

Next, we prove that $n_{k+1} \notin \{n_0, \ldots, n_k\}$. This will show that $\mathbf{N}$ is indeed infinite and hence that every natural number has a witness. From the infinity of $\mathbf{N}$ we get that every number has a successor; and from the fact that every number has a witness, we obtain that every number is distict from its successor.

3.17 Definition It's possible to extend the ordering $\prec$ between predicates to members of $\mathbf{N}$, by putting for x, y in $\mathbf{N}$: $x \prec y$ if and only if $\mathsf{VR}(P,x)$, $\mathsf{VR}(Q,y)$ and $P \prec Q$ (this is well-defined by BLV).

3.18 Lemma The ordering $\prec$ over $\mathbf{N}$ is total.

Proof. By induction on the generation of $\mathbf{N}$. ∎

3.19 Theorem If $\mathbf{N}x$, then $\forall y(\mathbf{N}y \wedge y \prec x \rightarrow y \neq x)$.

Proof. Here is the informal argument. Clearly, $n_0 \neq n_1$. Suppose for contradiction that for $j, k > 0$ and $j < k$, $n_j = n_k$. Then

$$\mathsf{VR}(\lambda y \mathbin{.} y = n_0 \vee \ldots \vee y = n_{j-1}) = \mathsf{VR}(\lambda y \mathbin{.} y = n_0 \vee \ldots \vee y = n_{k-1});$$

By BLV:

$$\forall y(y = n_0 \vee \ldots \vee y = n_{j-1} \leftrightarrow y = n_0 \vee \ldots \vee y = n_{k-1});$$

By first-order logic, in particular,

$$\forall y(y = n_k \rightarrow y = n_0 \vee \ldots \vee y = n_{j-1}),$$

i.e., $n_k = n_0 \vee \ldots \vee n_k = n_j$, against the inductive hypothesis. ∎

3.20 Theorem $\mathsf{Nn}(M) \wedge \mathsf{Sc}(M, N) \rightarrow M \neq N$

Proof. By induction on N, using the fact that $\forall M \neg \mathsf{Sc}(M, \mathsf{Z})$ and the previous theorem. ∎

4 The Consistency of $\mathcal{F}$

The issue to be considered in this section is the consistency of $\mathcal{F}$, which will be established by exhibiting a model, whose construction is due to Øystein Lynnebo. Lynnebo first observes that every witness of a natural number is the value range of a finite concept. Therefore, in order to satisfy axiom **F3**, it suffices to exhibit a model in which every concept of extensions of finite concepts has — in turn — an extension. Consider the (standard) model whose first-order domain is given by $V_{\omega+1}$ and in which for every $\alpha \leq \omega$, each concept $F \subseteq V_\alpha$ is assigned as its extension the corresponding set in $V_{\alpha+1}$.

Clearly, in such a model, each finite concept F is a subset of some V_k, and its extension therefore lies at the next level V_{k+1}. It follows that if C is a concept of extensions of finite concepts, two cases are possible: either (a) the concept C is itself finite, in which case it will have an extension in some V_{k+1}; or (b) the concept C is infinite, in which case its extension lies in $V_{\omega+1}$. It follows that any concept of extensions of finite concepts has itself an extension. This is enough to satisfy **F3**. Moreover, axiom **F2** is satisfied by the extensionality of sets, and axiom **F1** is satisfied because the model is standard. Our theory $\mathcal{F}$ is therefore consistent, indeed, equiconsistent (as one can easily see), with third-order arithmetic. It is perhaps worth noting — as John Burgess pointed out to us — that the system of Fine [2002] is likewise equiconsistent with third-order arithmetic.

We mention here that there is a weaker theory $\mathcal{F}_0$, in which the existential axiom **F3** is weakened to require only *finite* concepts of witnesses to have a value ranges. In particular, given a concept S, let $\mathsf{Fin}(S)$, abbreviate the second-order claim that S is Dedekind finite, i.e., $\neg\exists F[F$ is a proper injection of S into $S]$. Then $\mathcal{F}_0$ replaces **F3** by the axiom:

$$\mathsf{Fin}(\phi) \wedge \forall x(\phi(x) \rightarrow \exists M(\mathsf{Nn}(M) \wedge \mathsf{Wtn}(M, x))) \rightarrow \exists x\, \mathsf{VR}(\phi, x).$$

$\mathcal{F}_0$ is of interest because it is strong enough to derive arithmetic. The derivation of the Dedekind-Peano axioms given above goes through almost *verbatim* for $\mathcal{F}_0$, with only modifications necessary in the proof of Lemma 3.16 (where one needs to observe that if x is in $\mathbf{N}$, where $\mathsf{VR}(Q, x)$, and Q is a Dedekind-finite predicate of witnesses having number N, then the predicate $\lambda y \,.\, Qy \vee y = x$ is still Dedekind finite, and has number $\leq N + 1$; but then $\lambda y \,.\, Qy \vee y = x$ also applies to finitely many witnesses, and hence by the special existential axiom has a value range y in $\mathbf{N}$, witnessing a number $\leq M + 1$).

Moreover, $\mathcal{F}_0$ is easily seen to be consistent. In fact, if we content ourselves with a rough-and-ready interpretation, it becomes apparent that $\mathcal{F}_0$ can be embedded in a second-order version of PA, where the VR relation maps each finite set of natural numbers to (say) its standard code. This gives an interpretation of $\mathcal{F}_0$ relative to second-order arithmetic. On this interpretation, comprehension holds (**F2**), and moreover it is immediate that equi-extensional predicates have the same value-ranges (**F1**). (In particular, it follows that value ranges are unique.) Finally one proves by meta-theoretic induction on cardinality that every finite set of natural numbers has a value range, because every finite set of natural numbers is a finite set of witnesses (**F3**).

$\mathcal{F}_0$, we think, is of some independent interest largely because it is a system adequate for the derivation of arithmetic whose consistency strength is not any higher than that of the neo-logicist system of Hale and Wright. One might be tempted to think that since $\mathcal{F}_0$ builds the notion of finiteness into the axiom system, it is after all no surprise that arithmetic can be recovered. If this were the case, $\mathcal{F}_0$ would be yet another derivation of arithemetic from a theory of *finite sets*, of the kind surveyed (and criticized) by Parsons [1987]. Such a criticism would be only partially warranted. The finiteness assumptions certainly carries weight, as it is apparent upon inspection of the consistency argument for $\mathcal{F}_0$ given above, where every finite set (of witnesses or otherwise) has an extension. But the finiteness assumption itself — being at the second-order — plays no role in the derivation of arithmetic, as again becomes clear upon inspection of the proof.

It is worth mentioning some other systems that one might consider. First, there is a whole hierarchy of existential principles strictly between the characteristic axiom of $\mathcal{F}_0$ and that of $\mathcal{F}$. It is clear that some transfinite cardinals can be characterized in the second-order framework under consideration: for instance, "S is denumerably infinite" can be expressed in second order logic, and if "S has cardinality κ" can be expressed, then so can "S has cardinality κ^+". One can then envisage, for any such κ characterizable in second-order logic, an axiom to the effect that sets of witnesses of cardinality κ have value ranges.[7] Models for such intermediate systems have also been identified by Øystein Lynnebo: these are models of cardinality 2^κ in which every set of cardinality κ has a value range. Such a model has at

[7]Thanks to Sean Duggan-Ebels for suggesting this kind of intermediate axioms for consideration.

most $(2^\kappa)^\kappa = 2^\kappa$ subsets of cardinality κ, and using the axiom of choice one can obtain an injection of such subsets into the first-order objects of the model. The injection provides the needed interpretation for VR.

One might think that $\mathcal{F}$ could also be modified by strengthening the axioms. For instance, the following version of suggests itself: replace **F3** with an axiom such as:

Every concept of witnesses of (cardinal) numbers has a value range.

But as observed by John Burgess, such a system is inconsistent. First of all, call an object a *set* if it is the value range of a concept Y, and say that x is a member of y if x falls under Y. It follows that every set witnesses a cardinal: if y is a set, then comprehension gives the concept N of being the extension of a concept equinumerous to Y. But N is a (cardinal) number and y is a witness for N. Thus every set witnesses a cardinal, and if we assumed the strengthened axiom above it would follow that every concept of sets has an extension (or, as we also might say, every concept of sets *is*, in turn, a set). But then the concept R: "being a set that is not a self-member" would give Russell's paradox.

5 Hume's Principle

Perhaps the centerpiece of the neo-logicist program is what is known as "Frege's theorem," the result that arithmetic is derivable in second-order logic from *Hume's Principle*, the condition that the number of F equal the number of G if and only if there is a one-to-one correspondence between the F's and the G's. In the most highly developed neo-logicist approach, that of Hale and Wright [2001], Hume's Principle is *stipulated* as an axiom; its claimed importance is the role it plays in constituting the concept of number. The status of Hume's Principle in our approach is rather different: in fact it can be shown to fail in $\mathcal{F}_0$, which like the Hale and Wright system is consistent relative to second-order PA, as well as in the stronger $\mathcal{F}$.

To see why this is we need first to specify exactly what one means by Hume's Principle in the context of our systems. There are at least two natural readings of "The number of F's equals the number of G's if and only if there is a one-to-one correspondence between the F's and the G's." It is clear how to render the right-hand-side $F \approx G$ of the biconditional (see Def. 3.2). The issue arises with the left-hand-side. Recall that we have defined "M numbers the F's" (abbreviated $\mathsf{N}(F, M)$) as

$$\forall y(My \leftrightarrow \exists S(\mathsf{VR}(S, y) \wedge S \approx F)).$$

Then, "the number of F is the number of G" might taken to mean one of the following two statements:

1. $\exists M(\mathsf{N}(F,M) \wedge \mathsf{N}(G,M))$; or
2. $\forall M(\mathsf{N}(F,M) \leftrightarrow \mathsf{N}(G,M))$.

It turns out that it does not make a big difference, for the status of Hume's Principle, which version we take. To make the case, consider the first version, with the existential quantifier (the other is similar, but easier). It is easy to verify that the right-to-left direction holds. Assume $F \approx G$; by comprehension, let

$$M = \{y : \exists S(\mathsf{VR}(S,y) \wedge S \approx F)\};$$

since $F \approx G$ we also have

$$M = \{y : \exists S(\mathsf{VR}(S,y) \wedge S \approx G)\};$$

hence, $\exists M(\mathsf{N}(F,M) \wedge \mathsf{N}(G,M))$.

The converse, however, fails. Let us go through a proof attempt so that we can pinpoint exactly where the obstacle lies. Assume that for some M, we have both $\mathsf{N}(F,M)$ and $\mathsf{N}(G,M)$. In particular, it follows that

$$\forall y \Big[\exists S(\mathsf{VR}(S,y) \wedge S \approx F) \leftrightarrow \exists S'(\mathsf{VR}(S',y) \wedge S' \approx G)\Big].$$

For the proof to go through, we need to assume that $N \neq \emptyset$, for then there are S and y such that $\mathsf{VR}(S,y)$ and $S \approx F$. By the above, we also have that there is some S' such that $\mathsf{VR}(S',y)$ and $S' \approx G$. Since y is the value-range of both S and S', by BLV we have $S = S'$, which implies $F \approx G$, as desired.

However, if no $S \approx F$ has a value range (so that N is empty), the conclusion fails, then Hume's Principle no longer follows. Here's a counterexample: let A be an uncountable set of *urelements*, let $V_\omega[A]$ (the collection of all hereditarily finite sets over A) be the first-order domain, and as before let VR map finite sets of natural numbers to their codes. Now consider the two second-order concepts $\mathbb{N}$ and A, whose extensions are the natural numbers and the *urelements*, respectively. Since only finite sets of natural numbers have value ranges, no $S \approx A$ has a value range, and similarly no $S \approx \mathbb{N}$ has a value range. If, by comprehension, we let M_1 be the number of $\mathbb{N}$, and M_2 be the number of A, we have $M_1 = \emptyset = M_2$, and yet $A \not\approx \mathbb{N}$.

This gives the counterexample for $\mathcal{F}_0$; a similar argument can be given for $\mathcal{F}$. Assuming GCH to simplify things, let A be a set of cardinality $\aleph_2$ and consider the model having $V_{\omega+1}[A]$ as its first-order domain. As before, each concept $F \subseteq V_\omega[A]$ has an extension in $V_{\omega+1}[A]$. Now notice that since every $F \subseteq \mathbb{N}$ has an extension, $\mathcal{P}(\mathbb{N})$ is a concept, but no

$G \approx \mathcal{P}(\mathbb{N})$ has an extension. Similarly, no $G \approx A$ has an extension, whence the number of A is the same as the number of $\mathcal{P}(\mathbb{N})$, and yet $\mathcal{P}(\mathbb{N}) \not\approx A$.

So the implication from "the number of F equals the number of G" to "$F \approx G$" fails (although, as we have seen, the converse implication holds in $\mathcal{F}_0$). The reason this half of Hume's Principle fails is in a certain sense obvious: it is because we do not have enough value ranges around to witness the fact that $F \not\approx G$. Accordingly, if we had enough value ranges of the right kind, then this half of Hume's Principle would hold. In contrast, the other half of Hume's principle does not require special existential assumptions, as it deals only with the identity conditions of numbers.

There are two halves to Hume's Principle, just like there are two halves to Basic Law V. Whereas the left-to-right direction of Basic Law V expresses the injection of concepts into objects (value ranges), and the right-to-left direction expresses the identity conditions on those objects, Hume's principle expresses those relations with respect to a restricted range of objects, i.e., the numbers. In the case of Basic Law V, it is the left-to-right direction that is responsible for the contradiction. Notice that it is the same direction of Hume's Principle that is subject to our counter-example. The reason for this correlation is that once the implicit existential assumptions of Basic Law V (those than run afoul of Cantor's theorem) are drawn out and made explicit, those existential assumptions can be suitably weakened in such a way as to avoid the contradiction, maintaining Basic Law V in the form that we have given. But it is this weakening which cuts down on the available value ranges that also stands at the heart of our counter-example to Hume's principle. Thus what we see here is that once we isolate the mathematical roots of the contradiction, it is also possible, and perhaps desirable, to provide a Fregean foundation of arithmetic that does not assume Hume's Principle.[8]

We can further illustrate the characteristics of our system by comparing it to others that are closely related. One such approach is that of Demopoulos and Bell [1993]. They consider a second-order system with an explicit mapping e of concepts (of any order) into the objects. Russell's paradox is avoided in the system because the analogue of Frege's Basic Law V is not assumed in general, but only for the third-order *numerical concepts*, i.e., equivalence classes of concepts under the equinumerosity relation (Demopoulos and Bell refrain from calling such concepts "numbers," a name they instead reserve for the corresponding objects). Such an assumption is similar in inspiration to our extra-logical

[8]It is of some interest to conjecture whether Frege realized that there were embedded existential assumptions in Basic Law V. We know that he was familiar with Cantor's work, yet since it is only Hume's Principle that is needed for Frege's derivation of his version of PA, perhaps he might have thought that the existential assumptions were harmless and did not impact the status of Basic Law V as a logical law. He certainly did not think that existential assumptions implicit in abstraction principles undermined their explanatory efficacy, cf. the famous characterization of direction in terms of parallel lines in *Grundlagen*.

axiom **F3**, but whereas for Demopoulos and Bell numbers are tightly connected (perhaps identical) to certain objects satisfying Basic Law V, for us only certain second-order concepts (falling under the numbers) are witnessed among the objects. So, Demopoulos and Bell only assume Basic Law V for a restricted portion of the universe: in contrast, in our system, Basic Law V holds unconditionally for any concept having value ranges. As a result of this different approach, Demopoulos and Bell obtain the general validity of Hume's Principle, which is in fact entailed by Basic Law V, and they are precluded from driving a wedge between the resources necessary to obtain arithmetic and the further requirement expressed by Hume's Principle.

Another system of interest in the present context is that of Boolos [1996]; in this system, like ours, a counter-example to Hume's principle can be generated. Boolos considers a second-order language with a "number-of" operator defined on every concept, which is assumed to be *extensional*, in that if $P = Q$ then the number of P is the same as the number of Q (but no further assumptions are made — in particular, there is no requirement that equinumerous concepts be assigned the same number). Boolos then proceeds to give a counter-example to Hume's Principle by constructing a model in which two equinumerous sets (*viz.*, $\mathbb{N}$ and $\{2n : n \in \mathbb{N}\}$) are assigned two different objects as their "number." The counterexample does not go through in the present context, as value ranges (when they exist) are subject to Basic Law V. Our counter-example to Hume's Principle is quite different in nature: whereas in Boolos [1996] it is the right-to-left direction of the principle that fails, here it is the left-to-right one.[9] Thus, Boolos' counter-example is farther removed from the mathematical roots of the contradiction, it only addresses, in a somewhat contrived way, the issue of the identity conditions of the numbers.

6 Concluding remarks

To summarize, what we have accomplished is a reduction of arithmetic to a theory of extensions of concepts. As such, we have, in the context of a broadly Fregean program, detached extensionalism from logicism . Our claim is that Basic Law V is properly understood as part of the theory of extensions, and not as part of logicism. Hale and Wright, who take the logicist side, in effect agree with this, in that they reject Basic Law V in favor of Hume's Principle. (Hume's Principle, by our lights is not part of the theory of extensions.) Since both theories are consistent in the relevant sense and in both Peano Arithmetic is derivable, we can ask after the depth of insight provided by each system for the foundations of arithmetic. The problem with Hale and Wright's system, it seems to us, is that it gives no

[9] Boolos's argument is independent of the way we choose to reconstruct Hume's Principle with the universal or existential quantifier.

purchase on the underlying causes of the condradiction in Frege's system. The reason for this, we have argued, is that it does not reveal the underlying existential assumptions that specify the mathematical content of the system.

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References

Aldo Antonelli and Robert May, Frege's New Science, *Notre Dame Journal of Formal Logic*, 41(3):242–70, 2000, published 2002.

George Boolos, On the proof of Frege's theorem, in Adam Morton and Stephen Stitch, editors, *Benacerraf and his Critics*, pages 143–59, Blackwell, 1996, reprinted in Boolos [1998], pp. 275–90.

George Boolos, Is Hume's principle analytic?, in Richard Heck, editor, *Language, Thought and Logic: Essays in honour of Michael Dummett*, Oxford University Press, 1997.

George Boolos, *Logic, Logic, and Logic*, Harvard University Press, Cambridge, MA, 1998.

George Boolos and Richard Heck, Die Grundlagen der Arithmetik, §§82-3, in Matthias Schirn, editor, *The philosophy of mathematics today*, pages 407–28, Oxford University Press, 1998, reprinted in [Boolos, 1998, pp. 315–38].

William Demopoulos and John L. Bell, Frege's theory of concepts and objects and the interpretation of second-order logic, *Philosophia Mathematica (3)*, 1:139–56, 1993.

Fernando Ferreira and Kai F. Wehmeier, On the consistency of the Δ^1_1-CA fragment of Frege's *Grundgesetze*, *Journal of Philosophical Logic*, 31:301–11, 2002.

Kit Fine, *The Limits of Abstraction*, Oxford University Press, Oxford, 2002, 150pp.

Bob Hale and Crispin Wright, *The Reason's Proper Study. Essays toward a Neo-Fregean Philosophy of Mathematics*, Oxford-Clarendon Press, 2001.

Richard Heck, The consistency of predicative fragments of Frege's *Grundgesetze der Artithmetik*, *History and Philosophy of Logic*, 17:209–20, 1996.

Richard Heck, An introduction to Frege's theorem, *Harvard Review of Philosophy*, 7:56–73, 1999.

Harold Hodes, Logicism and the ontological commitment, *The Journal of Philosophy*, 81 (3):123–149, March 1984.

Charles Parsons, Developing arithmetic in set theory without infinity: Some historical remarks, *History and Philosophy of Logic*, 8:201–213, 1987.

Augustin Rayo, Frege's unofficial arithmentic, *The Journal of Symbolic Logic*, 67(4):1623–1638, Dec. 2002.

Kai F. Wehmeier, Consistent fragments of *Grundgesetze* and the existence of non-logical objects, *Synthése*, 121:309–28, 1999.

Crispin Wright, Is Hume's principle analytic?, *Notre Dame Journal of Formal Logic*, 40(1): 6–30, 1999.