Logicism without Logic

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Background

Can the notion of a truth of logic be explained otherwise than via the notion of provability?"

Boolos & Heck

Does Frege have metatheory?
In Fregean logic, is it possible to prove propositions that are about logic?

Answer: Yes! Frege's New Science
Die Grundlagen der Geometrie, II, sec. 3.

Applications of Logic with "pure logic" at its core but with non-logical axioms.

Frege's permutation method

The method is specified as an axiom of the New Science:

$\pi : L \to L$:

1. Double list of signs, each pairing respected logical type (names/names; concept-words/concepts words, etc)
2. Each sign has definite sense and reference (*no* reinterprtation).
3. Lists are related one-one (no sign is listed more than once on either side)
4. Signs "whose references belong to logic" are paired to themselves.

A proposition G is independent of a group Ω of propositions iff there is a permutation π such that $\pi(G)$ is false and $\Pi(\Omega)$ is true.

The case of geometry

Let Ω be the set of Euclidian axioms less the axiom of parallels, and G the axiom of parallels. Ω maps onto Ω' by a permutation that replaces expressions for points and lines of Euclidean plane geometry by expressions for the points and lines that lie in a fixed sphere, and maps "x is congruent to y" of plane geometry onto "x is taken into y by a linear transformations of the sphere into itself". As is well known, under this permutation the axiom of parallels fails, while all the other axioms of Euclidean geome try are still true; consequently, by Frege's method this "Hilbert" permutation establishes the independence of the parallel axiom.

The problem of the Logical Constants

Frege stipulates the logical constants (= classical truth-functional connectives and quantifiers) held invariant under permutation of L.

But for the method to be reliable, must know that these logical terms are *all* the logical terms

Otherwise there might be logical terms beside the ones stipulated that are mistakenly permuted, in which case some propositions might incorrectly turn out to be independent of other proposi tions; could have false positives.

A New Science characterization of the logical constants

Step 1 Assume a permutation π of the vocabulary of the language:

- atomic names for truth-values are mapped to atomic names for truth-values. (Truth-values names map to themselves or are switched.)
- If ϕ is a proposition Ca, let $\pi(\phi) = \pi(C)\pi(a)$.

Step 2 Identify a class of permutations, such that the logical truths are those propositions ϕ such that ϕ is mapped to the true if and only if the true is mapped to the true, and similarly for the false, given that truth-values are mapped to truth-values: where $d(\alpha)$ is the reference of α: $\{\phi | \forall \pi d(\pi(\phi)) = d(\pi(True))\}$

Step 3 Any class C of basic terms (including connectives, quantifiers, and variables of any order) will be a candidate class of logical constants if they are a largest class of terms such that every proposition in that vocabulary, if true, is logically true.

The New Science Project

Our goal is to make the project of Frege's New Science precise.

The project has two aspects:

1. To devise a system adequate for arithmetic
2. To explore the notion of logical truth, as characterized in the New Science, for this system.

Our focus today will be on (1).

The Language

We consider a second-order language $\mathcal{L}_2$ providing:

- individual variables and constants
- predicate variables and constants
- connectives and quantifiers
- a relation $\mathsf{VR}(P, x)$, x is the value range of P.

A model $\mathfrak{M}$ for $\mathcal{L}_2$ comprises a non-empty first-order domain D as well as, for each n, a non-empty collection of subsets of D^n (providing a range for the n-place second-order variables). Each n-place predicate constant (if there are any), is assigned a particular subset of D^n. Finally, corresponding to VR we have a function assigning a subset of D of cardinality ≤ 1 to each predicate variable P.

Numbers

Let $P \approx Q$ abbreviate the standard claim that there is a 1-1 correspondence between the P's and the Q's.

Define $\mathsf{N}(P, Q)$, "Q is the number of P's," if and only if Q is the concept 'y is the value-range of a concept $S \approx P$':

$$\forall y(Qy \leftrightarrow \exists S(\mathsf{VR}(S, y) \wedge P \approx S)).$$

We use second-order variables $N, M, P, \ldots$ for numbers, and by a slight abuse of notation, we also write $\mathsf{N}(N)$, "N is a number" as $\exists PN(P, N)$. We also write $\mathsf{Z}(N)$, "N is zero," if and only if $\exists P(\forall y \neg P(y) \wedge \mathsf{N}(P, N))$.

Define $\mathsf{Sc}(M, N)$, "N is the (immediate) successor of M," as follows:

$$\exists P \exists Q \exists z[\mathsf{N}(P, M) \wedge \mathsf{N}(Q, N) \wedge Qz \wedge \forall w(Pw \leftrightarrow Qw \wedge w \neq y)].$$

Witnesses

$\mathsf{Wtn}(N, x)$ if and only if $\exists P(\mathsf{N}(P, n) \wedge \mathsf{VR}(P, x)$.

The definition of *natural number* follows the standard (higher-order) inductive definition: N is natural number if and only if every predicate S which contains a witness for zero and such that if it contains a witness for M then it contains witnesses for any successors of M, also contains a witness for N:

$\mathsf{Nn}(N)$, "N is a natural number," iff:

$$\forall S[\exists y(\mathsf{Wtn}(\mathsf{Z}, y) \wedge Sy) \wedge \forall M(\mathsf{N}(M) \wedge \exists y(\mathsf{Wtn}(M, y) \wedge Sy)$$
$$\rightarrow \forall M'(\mathsf{Sc}(M, M') \rightarrow \exists y(\mathsf{Wtn}(M', y) \wedge Sy)))$$
$$\rightarrow \exists y(\mathsf{Wtn}(N, y) \wedge Sy)]$$

The theory $\mathcal{F}$

$\mathcal{F}$ will comprise the following *non-logical axioms*:

1. Any concept has at most one VR:
   $\forall P \forall x \forall y[\mathsf{VR}(P,x) \wedge \mathsf{VR}(P,y) \rightarrow x = y]$;
2. A version of Frege's "Basic Law V" (where $\overline{z} = z_1, \ldots z_n$):

$$\forall P \forall Q \forall x \forall y[\mathsf{VR}(P,x) \wedge \mathsf{VR}(Q,y) \rightarrow (\forall \overline{z}(P\overline{z} \leftrightarrow Q\overline{z}) \leftrightarrow x = y)];$$

3. A comprehension principle for any formula ϕ (with free parameters other than P allowed in ϕ): $\exists P \forall \overline{x}[P\overline{x} \leftrightarrow \phi(\overline{z})]$;
4. Special existential axioms providing for the existence of value ranges: for any formula ϕ,

$$\forall x(\phi(x) \rightarrow \exists M(\mathsf{Nn}(M) \wedge \mathsf{Wtn}(M,x))) \rightarrow \exists x\, \mathsf{VR}(\phi, x).$$

Induction

We introduce the predicate constant Z to denote the unique Q such that $\mathsf{Z}(Q)$.

The following induction principle is valid, for any formula $\Phi(P)$:

$$\Phi(\mathsf{Z}) \wedge \forall N(\mathsf{Nn}(N) \wedge \Phi(N) \rightarrow$$
$$\forall M(\mathsf{Sc}(N, M) \rightarrow \Phi(M))) \rightarrow \forall N(\mathsf{Nn}(N) \rightarrow \Phi(N)).$$

$M \leq N$ holds if and only if every set which contains a witness for M and is closed udner witnesses of successors, contains a witness for N

Dedekind-Peano axioms

Following Boolos, we consider the following axiomatization of arithmetic:

1. $\exists N(\mathsf{Z}(N) \wedge \mathsf{Nn}(N))$;
2. $\forall N \forall M(\mathsf{Nn}(N) \wedge \mathsf{Sc}(N, M) \rightarrow \mathsf{Nn}(M))$;
3. $\forall M \forall N_1 \forall N_2(\mathsf{Nn}(M) \wedge \mathsf{Sc}(M, N_1) \wedge \mathsf{Sc}(M, N_2) \rightarrow M = N)$;
4. $\forall N(\mathsf{Nn}(N) \rightarrow \exists M(\mathsf{Sc}(N, M) \wedge N \neq M))$;
5. $\forall N \forall M(\mathsf{Nn}(N) \wedge \mathsf{Z}(M) \rightarrow \neg \mathsf{Sc}(N, M))$;
6. $\forall M \forall N_1 \forall N_2(\mathsf{Nn}(N_1) \wedge \mathsf{Nn}(N_2) \wedge \mathsf{Sc}(N_1, M) \wedge \mathsf{Sc}(N_2, M) \rightarrow N_1 = N_2)$;
7. For every formula $\Phi(X)$ with the free second-order variable X:

$$\Phi(\mathsf{Z}) \wedge \forall M(\mathsf{Nn}(M) \wedge \Phi(M) \rightarrow \forall M'(\mathsf{Sc}(M, M') \rightarrow \Phi(M'))) \rightarrow \forall M(\mathsf{Nn}(M) \rightarrow \Phi(M')).$$

Special Witnesses

Let $\mathsf{Wtn}(\mathsf{Z}, \mathbf{z})$, where $\mathbf{z}$ is the value range of the empty predicate. Then $\mathbf{N}$ be the smallest predicate P containing $\mathbf{z}$ and satisfying the closure condition:

$$\forall Q \forall x \forall S \forall y [Px \wedge \mathsf{VR}(Q, x) \wedge \forall z (Sz \leftrightarrow (Qz \vee z = x)) \wedge \mathsf{VR}(S, y) \rightarrow Py].$$

$\mathbf{N} = \{n_0 \prec n_1 \prec \ldots\}$, where n_k is a special witness for K (n_k is the value range of the concept that applies to all previous witnesses):

$$\begin{aligned}
n_0 &= \mathsf{VR}(\lambda y \,\bullet\, y \neq y) = \mathbf{z} \\
n_1 &= \mathsf{VR}(\lambda y \,\bullet\, y \neq y \vee y = n_0) = \mathsf{VR}(\lambda y . y = n_0) \\
n_2 &= \mathsf{VR}(\lambda y \,\bullet\, y = n_0 \vee y = n_1) \\
&\vdots \\
n_{k+1} &= \mathsf{VR}(\lambda y \,\bullet\, y = n_0 \vee \ldots \vee y = n_k)
\end{aligned}$$

Main result

THEOREM: If $\mathbf{N}x$, then $\forall y(\mathbf{N}y \wedge y \prec x \rightarrow y \neq x)$.

Proof (Informal) Clearly, $n_0 \neq n_1$. Suppose for contradiction that for $j, k > 0$ and $j < k$, $n_j = n_k$. Then

$$\mathsf{VR}(\lambda y \,.\, y = n_0 \vee \ldots \vee y = n_{j-1}) = \mathsf{VR}(\lambda y \,.\, y = n_0 \vee \ldots \vee y = n_{k-1});$$

By BLV:

$$\forall y(y = n_0 \vee \ldots \vee y = n_{j-1} \leftrightarrow y = n_0 \vee \ldots \vee y = n_{k-1});$$

By first-order logic, in particular,

$$\forall y(y = n_{k-1} \rightarrow y = n_0 \vee \ldots \vee y = n_{j-1}),$$

i.e., $n_{k-1} = n_0 \vee \ldots \vee n_{k-1} = n_j$, against the inductive hypothesis.

$\mathsf{Nn}(M) \wedge \mathsf{Sc}(M, N) \rightarrow M \neq N$