Free Set Algebras Satisfying Systems of Equations

G. Aldo Antonelli[*]
Logic & Philosophy of Science
University of California
Irvine, California 92697-5100

Forthcoming in the Journal of Symbolic Logic

Abstract

In this paper we introduce the notion of a set algebra $\mathcal{S}$ satisfying a system $\mathcal{E}$ of equations. After defining a notion of freeness for such algebras, we show that, for any system $\mathcal{E}$ of equations, set algebras that are free in the class of structures satisfying $\mathcal{E}$ exist and are unique up to a bisimulation. Along the way, analogues of classical set-theoretic and algebraic properties are investigated.

In this note we introduce the notion of a set-algebra, i.e., an algebraic structure endowed with non-trivial set-theoretic properties. The motivation for doing so is provided by the desire further to investigate certain models for non-well-founded set theory introduced by the author in [2] and to account for the observation that such models seem to behave like "free" structures (in some sense) among those satisfying certain set-theoretic equations such as, e.g., $x = \{x\}$.

Non-well-founded models of set theory have been receiving increasing attention over the past few years, beginning with Aczel's seminal work [1]. For the most part, these models have been obtained as *largest* fixed points of certain operators induced by equations (see Barwise & Moss [4]). However, one can obtain models of non-well-founded set theory also as *smallest* fixed points, as it follows, for instance, from the approach of [2], where smallest fixed points are used to obtain models of ZFC minus Foundation satisfying certain equations. (For a further application of the method see [3].)

In order to make the analogy between models obtained via smallest fixed points and free structures precise, we need to carry out a "five-step program," by providing:

1. the right notion of set-theoretic *structure*;
2. a class of *maps* preserving that structure;
3. a notion of *freeness* that is appropriate for such structures and maps;
4. *existence* theorems for free structures;
5. *uniqueness* theorems for free structures, where 'uniqueness' is construed up to some notion of similarity.

This is analogous to what one usually finds in universal algebra, where, for instance, one might have:

---

[*] Thanks to an anonymous referee for helpful comments and criticisms.

structures = abelian groups
maps = homomorphisms
freeness = projective property
existence and uniqueness up to isomorphism

In our case we are interested in introducing set-algebras, along with a notion of *freeness* for such algebras relative to a system $\mathcal{E}$ of equations. It turns out that set algebras are very much like the kind of structures considered in universal algebra, with the added layer of complexity of having a relational structure (membership) imposed upon them. Since such a relational structure is required to be non-trivial (at least as complex as the hereditarily finite sets, as we will see), set-algebras cannot constitute a semi-variety (they are not axiomatizable by Horn clauses) — or, a fortiori, an algebraic (equational) variety. It follows, in particular, that the right notion of maps preserving set-theoretic structure is given by "set-like homomorphisms" ("set maps," for brevity), which, by the way, are formally equivalent to the p-morphisms of modal logic.

As mentioned, a centerpiece of any proposal for a notion of freeness for a given class of structures is given by existence and uniqueness results. In our case, results are given establishing, for each system $\mathcal{E}$ of equations, the existence of free set algebras satisfying $\mathcal{E}$ and their uniqueness up to a bisimulation. (The present approach is vaguely related to, but not quite the same in inspiration as either Mislove et al. [7] or Joyal & Moerdijk [6].) Pursuing the analogy with universal algebra somewhat further, it also turns out that the usual algebraic properties expressed in terms of homomorphisms have analogues expressed in terms of set maps.

On the set-theoretic side, we see that the approach of this paper can be extended to obtain, for each system $\mathcal{E}$ of equations, a model of full ZFC minus the axiom of Foundation satisfying those equations (see theorem 3.9). Such models differ from the structures considered in Aczel's theory of non-well-founded sets in that they can be more finely tuned to yield solutions only for some equations but not for others (solutions for system of equations must be bought "wholesale" in Aczel's system). This characterizes the structures obtained within the present approach as more conservative and having more concrete features than the ones obtained through the usual approach via largest fixed points.

Quite naturally, as we have seen, within the present framework, one considers the question of what set-theoretic properties set-algebras possess, as well as the separate question of which classic algebraic properties (from group theory or universal algebra) can be shown to obtain from these structures. These are interesting questions in and of themselves. But most interestingly perhaps, in this paper we also aim to begin the study of the way in which algebraic and set-theoretic properties are related, by asking, for instance, which set-theoretic properties can be expressed in purely algebraic terms involving only maps between structures. In particular we show that for a large class of set algebras (the ones that are set-theoretically well-founded) extensionality admits of a natural characterization in terms of set maps (see theorem 5.6 below).

1 Basic Definitions

1.1 DEFINITION A *set algebra* is a tuple $\mathcal{S} = (M, \in, +, s, 0)$ where M is a non-empty set, $\in$ a binary relation, $+$ a binary operator, s a unary operator, $0 \in M$, and $\mathcal{S}$ satisfies

Ax0 $x \notin 0$;

Ax1 $x \in s(x)$;

Ax2 if $x \in s(y)$ then $y \in s(x)$;

Ax3 if $x \in y$ and $y \in s(z)$ then $x \in z$;

Ax4 if $x \in s(y)$ and $y \in z$ then $x \in z$;

Ax5 $x \in y + z$ if and only if $x \in y$ or $x \in z$.

The intuitive interpretation is that 0 is an empty set, $x + y$ is the union of x and y, $s(x)$ is the singleton $\{x\}$, and $\in$ is, of course, membership. Observe that all the axioms, except the left-to-right half of Axiom 5, are Horn clauses. Axiom 5 appears to be crucial if we want set algebras to be at least as rich in structure as the hereditarily finite sets.

1.2 EXAMPLE The collection of all hereditarily finite (pure) sets is a set algebra. More in general, if we let $\mathcal{P}_{<\omega}(X)$ be the collection of all finite subsets of X, then any fixed point of $\mathcal{P}_{<\omega}$ is a set algebra.

A different notion of set algebras is can be found in Mislove et al. [7], and, in a somewhat different setting, in Joyal & Moerdijk [6]. Our definition differs from both of these.

It turns out that a sort of indiscernibility is definable in any set algebra.

1.3 DEFINITION For any x, y, put: $x \equiv y \iff x \in s(y)$.

1.4 LEMMA In any set algebra $\equiv$ is an equivalence relation which is also a congruence relation with respect to $\in$:

1. if $x \in y \equiv z$ then $x \in z$;
2. if $x \equiv y \in z$ then $x \in z$.

Proof. Reflexivity, symmetry and transitivity follow from Ax1, Ax2 and Ax3, respectively. Ax3 and Ax4 give the congruence. ■

1.5 DEFINITION A set algebra $\mathcal{S}$ is *weakly extensional* if it satisfies:

$$\forall z(z \in x \leftrightarrow z \in y) \Longrightarrow x \equiv y.$$

It is *strongly extensional*, or just *extensional*, if whenever $\forall z(z \in x \leftrightarrow z \in y)$, then $x = y$.

Obviously, strong extensionality implies weak extensionality.

1.6 LEMMA Let $\mathcal{S}$ be a weakly extensional set algebra. Then $\equiv$ is a congruence also with respect to $+$, i.e.: $x \equiv x'$ and $y \equiv y'$ imply $x + y \equiv x' + y'$.

Proof. Let $z \in x + y$. Then either $z \in x$ or $z \in y$. Assume the former (the latter being similar): then, since $\equiv$ is a congruence for $\in$, we have $z \in x'$, whence $z \in x' + y'$.

This shows that if $z \in x + y$ then $z \in x' + y'$. The converse implication can be established symmetrically. By weak extensionality, $x + y \equiv x' + y'$. ■

1.7 DEFINITION (GENERATED SET ALGEBRAS) Let $\mathcal{S}$ be a set algebra with support M, and $X \subseteq M$.

1. X is *closed* if $x, y \in X$ implies $s(x) \in X$ and $x + y \in X$.

2. The *closure* $\overline{X}$ of X is the smallest closed set $Y \supseteq X$.

3. $\mathcal{S}$ is *generated by* X if $M = \overline{X}$

1.8 DEFINITION (SET ALGEBRAS GENERATED BY E) Given a set $E = \{e_i : i \in I\}$, the set algebra generated by E and denoted by $\mathcal{S}[E]$, is defined as follows:

1. the support $M[E]$ is given by the collection of all terms over $E \cup \{0\}$;

2. s and $+$ are defined in the obvious way: $s^{\mathcal{S}[E]}(x) = s(x)$ and $x +^{\mathcal{S}[E]} y = x + y$;

3. we define the relation $x \in y$ by induction on the term y:
   (a) $x \in y$ never holds for $y \in E \cup \{0\}$;
   (b) $x \in s(y)$ holds if and only if x is y;
   (c) $x \in y + z$ holds if and only if either $x \in y$ holds or $x \in z$ holds.

Intuitively, we think of the objects in E as *urelements* or, equivalently, multiple copies of the empty set 0. Since set algebras carry a relational structure, it is not obvious that $\mathcal{S}[E]$ is a set algebra. We verify that this is the case below, along with a couple of other properties.

1.9 LEMMA Let $\mathcal{S}[E]$ be as above. Then:

1. $\mathcal{S}[E]$ is indeed a set algebra, i.e., axioms 0–5 are satisfied;

2. in $\mathcal{S}[E]$, the relation $\equiv$ is syntactical identity;

3. $\in$ is well-founded in $\mathcal{S}[E]$.

Proof. The verification of the axioms is immediate from the definition. Similarly, inspection of the definition gives that $\equiv$ is syntactical identity. All is left to show is that $\in$ is well founded. By induction on the generation of y we show that there are no descending $\in$-chains from y.

If $y \in E \cup \{0\}$ then y has no members, so the conclusion holds. Suppose y is $s(z)$; by inductive hypothesis, there are no infinite descending chains from z. If there were an infinite descending chain from y we would have:

$$\ldots \in x_3 \in x_2 \in x_1 \in s(z);$$

but then x_1 would be the same as z, so we would have an infinite chain from z. Finally, if there were an infinite descending chain from $y + z$ there would have to be an infinite descending chain from y or an infinite descending chain from z, against the inductive hypothesis. ■

1.10 DEFINITION (MAPS) Let $\mathcal{S}$ and $\mathcal{S}'$ be set algebras, and $h : M \to M'$ a map between their supports:

1. h is a *homomorphism* if and only if $h(0) = 0'$, h commutes with the operations of $\mathcal{S}$ (i.e., $+$ and s), and moreover if $x \in^{\mathcal{S}} y$ then $h(x) \in^{\mathcal{S}'} h(y)$.

2. h is an *epimorphism* if and only if h is a surjective homomorphism.

3. h is a *strong homomorphism* if and only if h is a homomorphism such that if $h(x) \in^{\mathcal{S}'} h(y)$ then $x \in^{\mathcal{S}} y$.

4. h is a *set-like homomorphism*, or simply a *set-map* if and only if h is a homomorphism for which the following 'back condition' holds: if $y \in h(x)$ then $y \equiv h(z)$ for some $z \in x$.

5. h is an *extensional set-like homomorphism*, or simply an *extensional set-map* if and only if the following version of the 'back condition' holds: if $y \in h(x)$ then $y = h(z)$ for some $z \in x$; notice that this can be naturally stated as the set identity $h(x) = \{h(y) : y \in x\}$.

The notion of a set-like homomorphism, whether extensional or not, is intermediate between the notions of a homomorphism and a strong homomorphism. In the case of purely relational structures such as the frames used in modal logic, extensional set-like homomorphisms are called p-morphisms. Conversely, it is interesting that in a set-theoretic context the 'back property' of p-morphisms can be naturally stated in the form of the set identity $h(x) = \{h(y) : y \in x\}$.

1.11 Definition (Bisimulation) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be set algebras with supports M_1 and M_2. A relation $R \subseteq M_1 \times M_2$ is a *bisimulation* if and only if it is a fixed point of the the following operator $(\cdot)^{\forall\exists}$:

$$R^{\forall\exists}(x, y) \iff (\forall u \in x)(\exists v \in y)\, R(u, v) \wedge (\forall v \in y)(\exists u \in x)\, R(u, v).$$

In the particular case where $\mathcal{S}_1 = \mathcal{S}_2$, it follows immediately that if R is an equivalence relation, then so is $R^{\forall\exists}$, and therefore, by induction, that the smallest bisimulation extending an equivalence relation R is also an equivalence relation. However, although bisimulations over some set algebra $\mathcal{S}$ are symmetric and transitive, they will not, in general, be equivalence relations.

It is also worth pointing out that this use of the term 'bisimulation' generalizes what is found in the literature. For instance, Barwise & Moss [4], define a bisimulation as a relation such that $R \subseteq R^{\forall\exists}$, but then proceed to consider the largest bisimulation, effectively restricting themselves to the largest fixed point of $(\cdot)^{\forall\exists}$. We, on the other hand, are interested in arbitrary fixed points of $(\cdot)^{\forall\exists}$, not necessarily the largest or the smallest.

1.12 Remark It follows from a folk theorem in the theory of monotone operators that if $R \subseteq R^{\forall\exists}$ then there is a bisimulation (indeed, a *smallest* bisimulation) extending R. We will make use of this fact below.

1.13 Definition Two set algebras $\mathcal{F}$ and $\mathcal{G}$ with supports F and G are *bisimilar* if and only if there is a bisimulation R with $\mathrm{dom}(R) = F$ and $\mathrm{rng}(R) = G$.

2 Systems of Equations

We begin by showing that that given a set algebra $\mathcal{S}$, any map $\alpha : E \to \mathcal{S}$ can be uniquely extended to a homomorphism $\beta : \mathcal{S}[E] \to \mathcal{S}$.

2.1 Lemma Let $\alpha : E \to \mathcal{S}$. Then, there is a unique homomorphism $\beta : \mathcal{S}[E] \to \mathcal{S}$ extending α.

Proof. We rehearse a standard argument. Let $\mathcal{S}$ be any set algebra, and $\alpha : E \to \mathcal{S}$ a map taking E into $\mathcal{S}$. define $\beta(x)$, for $x \in \mathcal{S}[E]$, by induction on x:

$$\begin{aligned} \beta(0) &= 0; \\ \beta(e) &= \alpha(e), \quad \text{for } e \in E; \\ \beta(s(x)) &= s(\beta(x)); \\ \beta(x + y) &= \beta(x) + \beta(y). \end{aligned}$$

Clearly β commutes with s and $+$ We need to show that if $x \in y$ holds in $\mathcal{S}[E]$ then $\beta(x) \in \beta(y)$ holds in $\mathcal{S}$. This can easily be shown by induction on y. For instance:

$$\begin{aligned} x \in s(y) &\Longrightarrow x = y \\ &\Longrightarrow \beta(x) = \beta(y) \\ &\Longrightarrow \beta(x) \in s(\beta(y)) \\ &\Longrightarrow \beta(x) \in \beta(s(y)), \end{aligned}$$

using the fact that $x \in s(y)$ holds in $\mathcal{S}[E]$ if and only if x is the same as y. So β is a homomorphism. Finally, β is unique, for if β' is another homomorphism with the same properties, then one shows $\beta(x) = \beta'(x)$ by induction on x. ■

2.2 DEFINITION Let $\alpha : E \to \mathcal{S}$. Then, $\hat{\alpha}$ denotes the unique homomorphism from $\mathcal{S}[E]$ into $\mathcal{S}$ that extends α.

2.3 DEFINITION (NORMAL TERMS) We inductively define the subclass of $M[E]$ comprising the *normal* terms as the smallest class of terms such that: (i) for any $x \in M[E]$, the term $s(x)$ is normal; and (ii) if x, y are normal then so is $x + y$.

It follows from the definition that if x is normal then x has members, and moreover it has no subterms of the form $y + e$ or $e + y$, for $e \in E$.

2.4 DEFINITION (SYSTEMS OF EQUATIONS) Let $E = \{e_i : i \in I\}$. Then:

1. An *equation* over E is a formal object of the form "$e \equiv t$," where $e \in E$ and $t \in M[E]$.
2. Given a set algebra $\mathcal{S}'$ (with support M'), a *solution* for an equation $e \equiv t$ (over E) is a function $f : E \to M'$ such that $\hat{f}(e) = \hat{f}(t)$.
3. A *system of equations over* E is a function $\mathcal{E} : E \to M[E]$ such that $\mathcal{E}(e)$ is a normal term for every e. If $\mathcal{E}(e) = t$ we will say that the equation $e \equiv t$ is in $\mathcal{E}$.

Observe that in Definition 2.4, part (2) the fact that $\hat{f}$ is a homomorphism extending f gives us that $\hat{f}(t(e_1, \ldots, e_n)) = t(f(e_1), \ldots, f(e_n))$.

We know from Lemma 2.1 that if $\mathcal{S}$ satisfies $\mathcal{E}$, then the function assigning to each $e \in E$ the corresponding object in $\mathcal{S}$ uniquely extends to a homomorphism from $\mathcal{S}[E]$ into $\mathcal{S}$. However, in general, such a homomorphism will not be set-like. As we will see, a little more work is needed to obtain a set map, even in the seemingly simple case in which $\mathcal{E}$ is empty.

We are now ready to introduce our notion of freeness. There are in general several notion of freeness available from algebra, and many turn out to be equivalent, e.g., in group theory. In an equational variety such as the theory of groups it is natural to define a group F to be free over a set $X \subseteq F$ of generators if any embedding of X into a group G uniquely extends to a homomorphism of F into G. In our case this definition needs to be generalized, since we have, in addition to the algebraic operations s and $+$, also a relational structure, viz. membership. As mentioned, this requires as a first approximation that we switch from homomorphisms to strong homomorphisms (the distinction being irrelevant in an equational context). But above all, since we are working relative to a system $\mathcal{E}$ of equations, we need to require that the mapping of the generators be a solution for $\mathcal{E}$. In return for weakening the hypothesis we obtain a strengthening of the freeness property: if the embedding of the generators is a solution for $\mathcal{E}$, then it extends not only to a homomorphism, but to a set-like homomorphism.

2.5 Definition (Freeness) Given a system $\mathcal{E}$ of equations over E, a class K of set algebras, and a map $\sigma : E \to \mathcal{S}$, we say that $(\mathcal{S}, \sigma)$ is *free on E in K relative to $\mathcal{E}$* if and only if for any map $\alpha : E \to \mathcal{S}' \in K$ that is a solution for $\mathcal{E}$, there is a unique set-like homomorphism $\beta : \mathcal{S} \to \mathcal{S}'$ such that for every $e \in E$, $\alpha(e) = \beta(\sigma(e))$.

$$\begin{tikzcd}
 & \mathcal{S} \arrow[dr, "\beta"] & \\
E \arrow[ur, "\sigma"] \arrow[rr, "\alpha"] & & \mathcal{S}'
\end{tikzcd}$$

In other words, $\mathcal{S}$ is free in K if and only if every solution $\alpha : E \to \mathcal{S}'$ extends to a unique set-map $\beta : \mathcal{S} \to \mathcal{S}'$.

2.6 Definition Given a system $\mathcal{E}$ of equations, $K(\mathcal{E})$ is the class of set-algebras $\mathcal{S}$ satisfying $\mathcal{E}$, i.e., such that there is a map $\alpha : E \to \mathcal{S}$ that is a solution for $\mathcal{E}$. Similarly, $\mathsf{Wext}(\mathcal{E})$ and $\mathsf{Ext}(\mathcal{E})$ are, respectively the class of all *weakly extensional* and *strongly extensional* set algebras satisfying $\mathcal{E}$.

When the set of equations $\mathcal{E}$ is empty, the above is the standard notion of freeness familiar, e.g., from group theory (see [8]).

3 Collapsing

3.1 Definition Given a system $\mathcal{E}$ of equations over E, a set algebra $\mathcal{S}$ and a function $f : E \to \mathcal{S}$, we define the relation $\simeq_{\mathcal{E}}$ over $\mathcal{S}$, as the smallest binary relation such that $\hat{f}(e) \simeq_{\mathcal{E}} \hat{f}(t)$ for any equation $e \equiv t$ in $\mathcal{E}$, and satisfying the clauses of reflexivity, transitivity, symmetry and monotony:

(ρ) $x \simeq_{\mathcal{E}} x$;

(τ) $x \simeq_{\mathcal{E}} y \wedge y \simeq_{\mathcal{E}} z \to x \simeq_{\mathcal{E}} z$;

(σ) $x \simeq_{\mathcal{E}} y \to y \simeq_{\mathcal{E}} x$;

(μ) for any x, y not both in $f[E] \cup \{0\}$, if $(\forall u \in x)(\exists v \in y)\, u \simeq_{\mathcal{E}} v$ and $(\forall v \in y)(\exists u \in x)\, u \simeq_{\mathcal{E}} v$, then $x \simeq_{\mathcal{E}} y$.

(Compare to [2, Definition 2.2, pp. 646–47], where $\mathcal{E}$ is assumed to contain only one equation.) Although the definition is completely general, in that it applies to arbitrary set algebras, we are going to be interested at first only in the case in which $\mathcal{S}$ is $\mathcal{S}[E]$. In particular given $\mathcal{E}$, the congruence relation $\simeq_{\mathcal{E}}$ on $\mathcal{S}[E]$ is to be understood to be relative to the identity map f on E.

It follows that when $\mathcal{E}$ is empty, $\simeq_{\mathcal{E}}$ is the smallest bisimulation on $\mathcal{S}[E]$ containing the identity relation. But in general, $\simeq_{\mathcal{E}}$ will not be a bisimulation since we have $e \simeq_{\mathcal{E}} t$ (when the equation '$e \equiv t$' is in $\mathcal{E}$), but e is memberless in $\mathcal{S}[E]$ — so that it can't be that for all $u \in t$ there is $v \in e$ such that $u \simeq_{\mathcal{E}} v$.

In [2] we also consider a directed relation $\preceq_{\mathcal{E}}$, defined as the smallest relation containing all the equations in $\mathcal{E}$ and closed under $(\rho), (\tau)$ and (μ) (but not (σ)). It is then possible to show that if $\mathcal{S}$ is well-founded then $\simeq_{\mathcal{E}}$ has the Church-Rosser property:

if $x \simeq y$ then for some z both $z \preceq_{\mathcal{E}} x$ and $z \preceq_{\mathcal{E}} y$.

However, the proof of the Church-Rosser property crucially employed the fact that only one equation $e \equiv t$ is contained in $\mathcal{E}$. When such an assumption is not available (as in the present context), it is easy to come up with a counter-example to the Church-Rosser property (even assuming well-foundedness). The question then is to find conditions sufficient in order that Knuth-Bendix completion of a system $\mathcal{E}$ of equations is successful.

3.2 CONJECTURE Knuth-Bendix completion is always successful.

Although no further use is made of this conjecture in this paper, we record it here as a problem to be taken up in future work.

Next, we show that clause (μ) is, to an extent, invertible: in particular, we want to show that if $z \in y$ and $y \simeq_{\mathcal{E}} y'$, where y' is a set (i.e., not a memberless atom — or, equivalently, not in E), then there exists $z' \in y'$ such that $z \simeq_{\mathcal{E}} z'$. Clearly, this follows from the following lemma.

3.3 LEMMA (THE "CONVERSE OF μ") Let y' be a set, i.e., not in E. Then:

(a) If $z \in y$ and $y \simeq_{\mathcal{E}} y'$ then there exists $z' \in y'$ such that $z \simeq_{\mathcal{E}} z'$;

(b) If $z \in y$ and $y' \simeq_{\mathcal{E}} y$ then there exists $z' \in y'$ such that $z \simeq_{\mathcal{E}} z'$.

Proof. We establish (a) and (b) by simultaneous induction on $\simeq_{\mathcal{E}}$.

Suppose $y \simeq_{\mathcal{E}} y'$ because the equation $y \equiv y'$ is in $\mathcal{E}$; then $y \in E$ and $z \in y$ never holds in $\mathcal{S}[E]$; so (a) is vacuous. For (b), suppose $y' \simeq_{\mathcal{E}} y$ because the equation $y' \equiv y$ is in $\mathcal{E}$, then y' cannot be a set, so this case is also vacuous.

Suppose $y \simeq_{\mathcal{E}} y'$ by (ρ). Then $y = y'$, and for (a) to hold suffices to take $z' = z$. Part (b) is similar.

Suppose $y \simeq_{\mathcal{E}} y'$ by (τ). Then there exists y'' such that $y \simeq_{\mathcal{E}} y'' \simeq_{\mathcal{E}} y'$. By inductive hypothesis on (a), $z \simeq_{\mathcal{E}} z'' \in y'' \simeq_{\mathcal{E}} y'$, for some z''. Again by inductive hypothesis, on (a), $z'' \simeq_{\mathcal{E}} z' \in y'$, for some z'. So $z \simeq_{\mathcal{E}} z' \in y'$, and (a) holds. Part (b) is similar.

Suppose $y \simeq_{\mathcal{E}} y'$ by (σ). Then $y' \simeq y$, and we use the inductive hypothesis on (b) to show that (a) holds. Part (b) is likewise obtained by inductive hypothesis on (a).

Finally, suppose $y \simeq_{\mathcal{E}} y'$ by (μ). Then obviously there is $z' \in y'$ such that $z \simeq_{\mathcal{E}} z'$. So (a) holds, and part (b) is equally immediate. ■

3.4 DEFINITION Given a set-algebra $\mathcal{S}$, a system $\mathcal{E}$ of equations over E, and a map $f : E \to \mathcal{S}$, we define the quotient $\mathcal{S}/\simeq_{\mathcal{E}}$ of $\mathcal{S}$ with respect to $\simeq_{\mathcal{E}}$. The support of the quotient is the collection $M/\simeq_{\mathcal{E}}$ of all equivalence classes $[x] = \{y : y \simeq_{\mathcal{E}} x\}$, and the empty set of the quotient is given by $[0]$. Moreover:

1. $[x] \in [y]$ iff in $\mathcal{S}[E]$ we have for some z either: (i) $x \simeq_{\mathcal{E}} z$ and $z \in y$; or (ii) $x \in z$ and $z \simeq_{\mathcal{E}} y$;
2. $s([x]) = [s(x)]$;
3. $[x] + [y] = [x + y]$.

It is perhaps interesting to remark that we might have not required set algebras to have a distinguished element 0. In which case $\mathcal{S}[E]$ would have been the usual collection of hereditarily finite sets over E (where the e's are regarded as multiple empty sets). But notice that then $\mathcal{S}[E]/\simeq_{\mathcal{E}}$ would not have contained any memberless object: all sets in it would have members: an unusual, but not in itself pathological situation.

We now proceed to give the salient properties of the quotients thus obtained.

3.5 Lemma If $x \simeq_{\mathcal{E}} 0$ then $x = 0$.

Proof. By induction on $\simeq_{\mathcal{E}}$. The base case is vacuous: since 0 is not a normal term, $x \equiv 0$ cannot be an equation in $\mathcal{E}$. The cases for reflexivity, symmetry, and transitivity are immediate. So assume $x \simeq_{\mathcal{E}} 0$ by (μ). Then, since $0 \in E \cup \{0\}$, x must have members; but then 0 must have members as well. So this case is vacuous too. ■

We now verify that the operations defined on the quotient structure are independent of the representatives:

3.6 Lemma Suppose $x \simeq_{\mathcal{E}} x'$ and $y \simeq_{\mathcal{E}} y'$. Then the following hold in $\mathcal{S}[E]$:

1. $[x] \in [y]$ only if $[x'] \in [y']$;
2. $s([x]) = s([x'])$;
3. $[x] + [y] = [x'] + [y']$.

Proof. Suppose $[x] \in [y]$. Then for some z, we have $x \simeq_{\mathcal{E}} z$ and $z \in y$. In particular, since y has members, it is a set. Using the converse of (μ) we reason as follows:

$$\begin{aligned}
[x] \in [y] &\implies x \simeq_{\mathcal{E}} z \in y \\
&\implies x' \simeq_{\mathcal{E}} x \simeq_{\mathcal{E}} z \in y \\
&\implies x' \simeq_{\mathcal{E}} z \in y \\
&\implies x' \simeq_{\mathcal{E}} z \in y \simeq_{\mathcal{E}} y' \\
&\implies x' \simeq_{\mathcal{E}} z \simeq_{\mathcal{E}} z' \in y' \\
&\implies x' \simeq_{\mathcal{E}} z' \in y' \\
&\implies [x'] \in [y'].
\end{aligned}$$

This establishes the first part of the lemma.

Now assume $u \in s(x)$; then $u = x$, and since $x \simeq_{\mathcal{E}} x'$, also $u \simeq_{\mathcal{E}} x'$. Since $x' \in s(x')$, this shows

$$(\forall u \in s(x))(\exists v \in s(x'))\, u \simeq_E v.$$

The converse is obtained similarly, whence $s(x) \simeq_{\mathcal{E}} s(x')$ by (μ). In turn, this implies $s([x]) = [s(x)] = [s(x')] = s([x'])$, as desired.

Finally, assuming $u \in x + y$, we have $u \in x$ or $u \in y$, whence either $\exists v[u \simeq_{\mathcal{E}} v \wedge v \in x']$ or $\exists v[u \simeq_{\mathcal{E}} v \wedge v \in y']$, which implies $\exists v[u \simeq_{\mathcal{E}} v \wedge v \in x' + y']$. This shows

$$(\forall u \in x + y)(\exists v \in x' + y')\, u \simeq_E v.$$

since the converse is obtained similarly, this implies $x + y \simeq_{\mathcal{E}} x' + y'$. As before, in turn, this implies $[x] + [y] = [x'] + [y']$. ■

3.7 Theorem Let $\mathcal{S}[E]/\simeq_{\mathcal{E}}$ be the quotient of $\mathcal{S}[E]$ with respect to $\simeq_{\mathcal{E}}$. Then $\mathcal{S}[E]/\simeq_{\mathcal{E}}$ is a strongly extensional set algebra satisfying all equations in $\mathcal{E}$.

Proof. First we show that $\mathcal{S}[E]/\simeq_{\mathcal{E}}$ is a set algebra. For this, we need to check the axioms in turn. Ax0 follows because if $[x] \in [0]$ then $x \simeq_{\mathcal{E}} x' \in 0$, which is impossible. Ax1 is equally immediate: since $\simeq_{\mathcal{E}}$ is reflexive, we have $x \simeq_{\mathcal{E}} x \in s(x)$, whence $[x] \in s([x])$.

For Ax2, assume $[x] \in s([y])$, i.e., $[x] \in [s(y)]$; then $x \simeq_{\mathcal{E}} z \in s(y)$, for some z. It follows that $z = y$, whence $y \simeq_{\mathcal{E}} x \in s(x)$, and by definition $[y] \in s([x])$.

For Ax3, assume $[x] \in [y]$ and $[y] \in [s(z)]$. Then there are x' and y' such that $x \simeq_{\mathcal{E}} x' \in y$ and $y \simeq_{\mathcal{E}} y' \in s(z)$. In particular, $y' = z$ so that

$$x \simeq_{\mathcal{E}} x' \in z;$$

by the converse of (μ), $x' \simeq_{\mathcal{E}} z' \in z$, and by transitivity $x \simeq_{\mathcal{E}} z' \in z$. By definition, $[x] \in [z]$. Ax4 is similar.

For Ax5, assume $[x] \in [y + z]$; then there is x' such that $x \simeq_{\mathcal{E}} x'$ and either $x' \in y$ or $x' \in z$; if the former, $[x] \in [y]$, and if the latter $[x] \in [z]$. Conversely, if $[x] \in [y]$ then $x \simeq_{\mathcal{E}} x' \in y$, whence $x \simeq_{\mathcal{E}} x' \in y + z$ and $[x] \in [x + y]$. So all axioms holds and $\mathcal{S}[E]/ \simeq_{\mathcal{E}}$ is a set algebra.

Now we show that $\mathcal{S}[E]/ \simeq_{\mathcal{E}}$ is strongly extensional. Assume that for all z, $[z] \in [x]$ if and only if $[z] \in [y]$. We need to show that $[x] = [y]$, i.e., $x \simeq_{\mathcal{E}} y$.

Assume $u \in x$; then also $[u] \in [x]$ by reflexivity so that $[u] \in [y]$ by hypothesis. It follows that there exists a v such that $u \simeq_{\mathcal{E}} v$ and $v \in y$. This shows

$$(\forall u \in x)(\exists v \in y)\, u \simeq_{\mathcal{E}} v;$$

Since the converse can be established symmetrically, we have $x \simeq_{\mathcal{E}} y$, as desired.

Finally, we need to show that all equations in $\mathcal{E}$ are satisfied in $\mathcal{S}[E]/ \simeq_{\mathcal{E}}$. It is clear that if $e \equiv t$ is an equation in $\mathcal{E}$, then $e \simeq_{\mathcal{E}} t$, so that $[e] = [t]$. Moreover, a solution can be obtained explicitly as follows. We need a map $f : E \to \mathcal{S}[E]/ \simeq_{\mathcal{E}}$ such that for each equation $e \equiv t(e_1, \ldots, e_n)$ it holds that $f(e) = t(f(e_1), \ldots, f(e_n))$. We can take such a map to be $f(e) = [e]$.

We know that for any x, $s([x]) = [s(x)]$ and for any x and y, $[x] + [y] = [x + y]$. Therefore, inductively, we can show that for any term $t(e_1, \ldots, e_n)$, also $[t(e_1, \ldots, e_n)] = t([e_1], \ldots, [e_n])$. In particular, then we have $[e] = [t(e_1, \ldots, e_n)] = t([e_1], \ldots, [e_n])$, which shows that the map f defined above is indeed a solution. ■

The previous result shows that collapsing with respect to a congruence relation such as $\simeq_{\mathcal{E}}$ preserves a certain amount of set-theoretic structure; it preserves, in particular, the hereditarily finite structure given by the axioms 0–5. In the remainder of this section we show that a lot more structure is preserved by the collapsing.

3.8 Definition Let $\mathcal{S}$ be a set-algebra and $\mathcal{E}$ a system of equations; then $\mathcal{E}$ is *definable* over $\mathcal{S}$ (from parameters), if there is a function $f : E \to \mathcal{S}$ such that $\mathcal{E}$ as a function is definable from parameters, in the following sense: there is a formula $\phi(x, y)$ possibly involving parameters such that $\phi(\hat{f}(e), \hat{f}(t))$ holds in $\mathcal{S}$ if and only if the equation $e \equiv t$ is in $\mathcal{E}$.

Recall that in Definitions 3.1 and 3.4, the congruence relation $\simeq_{\mathcal{E}}$ and the corresponding quotient were defined in the general case of any set algebra $\mathcal{S}$ having a map $f : E \to \mathcal{S}$. With this in mind, we can now state the following result.

3.9 Theorem Let $\mathcal{S}$ be a set algebra which is, in fact, a model of ZFC, and suppose $\mathcal{E}$ is a system of equations definable over $\mathcal{S}$. Then $\mathcal{S}/ \simeq_{\mathcal{E}}$ is a model of ZFC minus the axiom of Foundation.

Proof. The argument is given in every detail in [2, pp. 655–77]. Here we only say something about the crucial axioms, replacement and choice. For replacement, the idea is that if $\mathcal{E}$ is definable, then so is $\simeq_{\mathcal{E}}$ (by transfinite recursion), so that for any ϕ we can find a formula $\phi^{\mathcal{E}}$ which says, in $\mathcal{S}$, that ϕ holds in $\mathcal{S}/\simeq_{\mathcal{E}}$. Using this and replacement in $\mathcal{S}$ one can show that replacement holds in the quotient.

For choice, one has to look at the stages in the ordinal definition of $\simeq_{\mathcal{E}}$ and define a choice function (in the sense of $\mathcal{S}/\simeq_{\mathcal{E}}$) for the equivalence class of a given set x using a choice function for x in $\mathcal{S}$ and a well-ordering of x. ■

Consider for instance a model $\mathcal{S}$ of ZFC having a unique urelement e, and the system $\mathcal{E}$ comprising the single equation $e \equiv s(e)$. Clearly $\mathcal{E}$ is definable in the model, and therefore the associated quotient $\mathcal{S}/\simeq_{\mathcal{E}}$ is a model of ZFC minus foundation satisfying the equation $e \equiv s(e)$, and consequently also all the identities:

$$\begin{aligned} e &= \{e\} \\ &= \{\{e\}\} \\ &= \{e, \{e\}\} \\ &= \{e, \{e\}, \{\{e\}\}\} \\ &= \ldots \end{aligned}$$

Such a model differs from models of Aczel's Anti-Foundation Axiom (AFA) in that only the single equation $e \equiv s(e)$ is guaranteed to have a solution, whereas AFA guarantees the existence of solutions for *all* systems of equations. As we will see below, in the case where $\mathcal{S}$ is $\mathcal{S}[E]$, the quotient $\mathcal{S}/\simeq_{\mathcal{E}}$ is a free set algebra among those satisfying $e \equiv s(e)$; but where $\mathcal{S}$ is a model of full ZFC the quotient is much too big to be free: still, intuitively, it appears to be very much like the smallest among the extensions of its pure part containing a solution to the equation $e \equiv s(e)$ and satisfying the same set construction principles. It is difficult to see how to make this intuition precise. One would need a clear account of what it means for two set algebras to "satisfy the same set construction principles" and then use such an account to define, for any set algebra $\mathcal{S}$, an embedding of $\mathcal{S}/\simeq_{\mathcal{E}}$ into any given extension $\mathcal{S}'$ of the pure part of $\mathcal{S}$ such that $\mathcal{S}'$ "satisfies the same set construction principles" as $\mathcal{S}$ and contains a solution for the equation $e \equiv s(e)$.

If this can be carried out — a project to be taken up in future work — then the differences between the present approach and AFA would become apparent: whereas AFA commits us to the existence of "all" non-well-founded sets — wholesale, as it were — the kind of quotients considered here give us a method to obtain, for any universe of *pure* sets, the smallest extension of that universe that still satisfies the same set extension principles and simultaneously contains solutions for all the equations in a system.

4 Freeness

We now set out to establish that $\mathcal{S}[E]/\simeq_{\mathcal{E}}$ is free in the class of all weakly extensional set algebras satisfying $\mathcal{E}$, relative to $\mathcal{E}$. For this, we need to obtain first a few intermediate results. In particular, given such a weakly extensional set algebra $\mathcal{S}$ and a solution α for $\mathcal{E}$, we define a map $\gamma : \mathcal{S}[E] \to \mathcal{S}$ and investigate its properties.

4.1 DEFINITION Let $\mathcal{S}$ be a weakly extensional set algebra, $\mathcal{E}$ a system of equations over E and $\alpha : E \to \mathcal{S}$ a solution for $\mathcal{E}$, and $\mathcal{S}[E]$ the set algebra generated by E. We define a map $\gamma : \mathcal{S}[E] \to \mathcal{S}$ as follows:

$$\gamma(0) = 0;$$

$$\begin{array}{rcl}\gamma(e) & = & \alpha(e), \quad \text{for } e \in E; \\ \gamma(s(x)) & = & s(\gamma(x)); \\ \gamma(x+y) & = & \gamma(x) + \gamma(y).\end{array}$$

Since this is the same definition as for the function β is Lemma 2.1, we know that γ is a homomorphism. Nonetheless, we rehearse some the its properties in the following two results.

4.2 LEMMA Let α and γ be as in Definition 4.1, and $t(e_1, \ldots, e_n)$ be a term in $\mathcal{S}[E]$. Then $\gamma(t) = t(\alpha(e_1), \ldots, \alpha(e_n))$

Proof. Immediate, by induction on t. ■

4.3 LEMMA Let γ be as in Definition 4.1. If $x \in y$ holds in $\mathcal{S}[E]$ then $\gamma(x) \in \gamma(y)$ holds in $\mathcal{S}$.

Proof. By induction on y. If $y \in E \cup \{0\}$ then the lemma hods vacuously. If $x \in s(z)$ then $x = z$, so $\gamma(x) = \gamma(z)$ so $\gamma(x) \in s(\gamma(z)) = \gamma(s(z))$. Finally, if $x \in y + z$ then either $x \in y$ or $x \in z$. Assume the former (the latter being similar). Then by inductive hypothesis $\gamma(x) \in \gamma(y)$, whence $\gamma(x) \in \gamma(y) + \gamma(z) = \gamma(y + z)$. ■

4.4 LEMMA Let γ be as in Definition 4.1, and t be a normal term of the form $t(e_1, \ldots, e_n)$. Then if $x \in \gamma(t)$, there is $y \in t$ such that $x \equiv \gamma(y)$.

Proof. By induction on t. The case $t \in E \cup \{0\}$ is vacuous, since then t is not normal. Suppose $t = s(y)$; then $x \in \gamma(t) = \gamma(s(y)) = s(\gamma(y))$, which implies $x \equiv \gamma(y)$, by definition of $\equiv$. The last case is when $t = y + z$: then if $x \in \gamma(y + z)$ we have either $x \in \gamma(y)$ or $x \in \gamma(z)$. Assume the former (the latter being similar). Since t is normal, so is y; by inductive hypothesis, $x \equiv \gamma(u)$ for some $u \in y$. But then also $u \in y + z$, as desired. ■

4.5 LEMMA Let γ be as in Definition 4.1. If $v \in \gamma(y)$ then there is $x \simeq_{\mathcal{E}} y$ and $u \in x$ such that $\gamma(u) \equiv v$.

Proof. Let α be the given solution for $\mathcal{E}$. We proceed by induction on y:

1. Suppose $y = 0$. then $\gamma(y) = 0$ as well, and the lemma holds vacuously.
2. Suppose $y \in E$. Then there is an equation $y \equiv t$ in $\mathcal{E}$, and we can take $x = t$. Indeed, $x \equiv y$, and moreover, since α is a solution, $\gamma(y) = \alpha(y) = t(\alpha(\overline{e})) = \gamma(t)$. In particular, $v \in \gamma(t)$, and since t is normal, by Lemma 4.4 we can find $u \in t$ such that $\gamma(u) \equiv v$.
3. Suppose $y = s(w)$. Then $\gamma(y) = \gamma(s(w)) = s(\gamma(w))$. So if $v \in \gamma(y)$, also $v \in s(\gamma(w))$. By definition of $\equiv$, we have $v \equiv \gamma(w)$. We can then take $x = s(w)$ and $u = w$, for then $x \simeq_{\mathcal{E}} y$ by (ρ), and $\gamma(u) \equiv v$.
4. Suppose $y = x_1 + x_2$. Then $v \in \gamma(x_1)$ or $v \in \gamma(x_2)$. Assume the former (the latter being similar). By inductive hypothesis, there are $x^* \simeq_{\mathcal{E}} x_1$ and $u \in x^*$ such that $\gamma(u) \equiv v$. Then we have that $(x_1 + x_2) \simeq_E (x^* + x_2)$ by (μ), and $u \in (x^* + x_2)$ as desired.

■

4.6 LEMMA If $x \simeq_{\mathcal{E}} y$ then $\gamma(x) \equiv \gamma(y)$.

Proof. By induction on $\simeq_{\mathcal{E}}$.

1. Suppose $x \simeq_{\mathcal{E}} y$ because the equation $x \equiv y$ is in $\mathcal{E}$. Then $\gamma(x) = \alpha(x) = y(\alpha(\overline{e})) = \gamma(y(\overline{e}))$, as desired.
2. Suppose $x \simeq_{\mathcal{E}} y$ by reflexivity, symmetry, or transitivity: then the conclusion is either immediate (for (ρ)), or follows from the inductive hypothesis.
3. Suppose $x \simeq_{\mathcal{E}} y$ by (μ). In particular, then both x and y are sets. We want to show that $\gamma(x) \equiv \gamma(y)$. Since $\mathcal{S}$ is weakly extensional, it suffices to establish
$$\forall z(z \in \gamma(x) \leftrightarrow z \in \gamma(y)).$$
So let $z \in \gamma(x)$. By lemma 4.5, there are $x' \simeq_{\mathcal{E}} x$ and $u \in x'$ such that $z \equiv \gamma(u)$. Since $u \in x'$ and $x' \simeq_{\mathcal{E}} x \simeq_{\mathcal{E}} y$, by the converse of (μ) there is $v' \in y$ such that $v \simeq_{\mathcal{E}} u$. By the inductive hypothesis, $\gamma(v) \equiv \gamma(u)$. Since $\gamma(u) \equiv z$, we have $\gamma(v) \equiv z$ by transitivity of $\equiv$.
Now we know that $v \in y$, so by Lemma 4.3 also $\gamma(v) \in \gamma(y)$. Since $\equiv$ is a congruence and $z \equiv \gamma(v)$, also $z \in \gamma(y)$.
This shows that if $z \in \gamma(x)$ then $z \in \gamma(y)$, and since the converse implication can be established symmetrically, weak extensionality yields $\gamma(x) \equiv \gamma(y)$, as desired.

■

4.7 THEOREM Let ι be the map on E such that $\iota(e) = [e]$. Then $(\iota, \mathcal{S}[E]/\simeq_{\mathcal{E}})$ is free on E in $\mathsf{Wext}(\mathcal{E})$ relative to $\mathcal{E}$.

Proof. Let $\mathcal{S}$ be any set algebra satisfying $\mathcal{E}$, and let $\alpha : E \to \mathcal{S}$ be a solution for $\mathcal{E}$. We need to show that there is a unique set-like homomorphism $\beta : \mathcal{S}[E]/\simeq_{\mathcal{E}} \to \mathcal{S}$ such that $\alpha(e) = \beta(\iota(e)) = \beta([e])$. Define β by putting:
$$\beta([x]) = \gamma(x),$$
where γ is the map of Definition 4.1. By Lemma 4.6, the definition of β is independent of the representative.

It is immediately apparent from the definition that $\alpha(e) = \beta(\iota(e))$ (for each $e \in E$), and $\beta([0])$ is the empty set of $\mathcal{S}$. To see that β commutes with s and $+$:
$$\begin{aligned} \beta([x] + [y]) &= \beta([x + y]) \\ &= \gamma(x + y) \\ &= \gamma(x) + \gamma(y) \\ &= \beta([x]) + \beta([y]); \end{aligned}$$
the case for successor is similar. Next, we need to show that if $[x] \in [y]$ holds in $\mathcal{S}[E]/\simeq_{\mathcal{E}}$, then $\beta([x]) \in \beta([y])$ holds in $\mathcal{S}$. Using Lemmas 4.3 and 4.6, we have:
$$\begin{aligned} [x] \in [y] &\implies x \simeq_{\mathcal{E}} z \in y \\ &\implies \gamma(x) \equiv \gamma(z) \in \gamma(y) \\ &\implies \gamma(x) \in \gamma(y) \\ &\implies \beta([x]) \in \beta([y]). \end{aligned}$$

So β is a homomorphism. To show that β is a set map suffices to verify the 'back condition:' if $\beta([x]) \in \beta([y])$ then there is a z such that $[z] \in [x]$ and $\beta([x]) \equiv \beta([z])$.

Assume $\beta([x]) \in \beta([y])$; then $\gamma(x) \in \gamma(y)$, and by Lemma 4.5 there are $w \simeq_{\mathcal{E}} y$ and $z' \in w$ such that $\gamma(z') \equiv \gamma(y)$. By the converse of (μ), there is $z \in y$ such that $z \simeq_{\mathcal{E}} z'$; by Lemma 4.6, also $\gamma(z) \equiv \gamma(z')$ whence by transitivity $\gamma(z) \equiv \gamma(y)$. But now $z \in y$ implies $[z] \in [y]$, and $\gamma(z) \equiv \gamma(y)$ implies $\beta([z]) \equiv \beta([y])$, as desired.

All is left to show is that β is a unique set-like homomorphism. Indeed, we show that β is a unique homomorphism with the desired properties. Let β^* be a homomorphism extending α. We show $\beta^*([x]) = \beta([x])$ by induction on x:

$$\begin{array}{rcl}
\beta^*([0]) = & 0 & = \beta([0]) \\
\beta^*([e]) = & \alpha(e) & = \beta([e]) \\
\beta^*([x]+[y]) & = & \beta^*([x]) + \beta^*([y]) \\
 & = & \beta([x]) + \beta([y]) \\
 & = & \beta([x]+[y]) \\
\beta^*(s([x])) & = & s(\beta^*([x])) \\
 & = & s(\beta([x])) \\
 & = & \beta(s([x])).
\end{array}$$

■

Given that $\mathcal{S}[E]/\simeq_{\mathcal{E}}$ is strongly extensional, inspection of the argument of this section establishes the following result.

4.8 Theorem Let ι be the map on E such that $\iota(e) = [e]$, and $\mathcal{S}$ a strongly extensional set-algebra satisfying $\mathcal{E}$. Then any solution $\alpha : E \to \mathcal{S}$ for $\mathcal{E}$ can be extended to a unique extensional set-map $\beta : \mathcal{S}[E]/\simeq_{\mathcal{E}} \to \mathcal{S}$ such that $\alpha(e) = \beta(\iota(e))$ for $e \in E$.

Proof. The hypothesis of strong extensionality allows us to replace $\equiv$ by $=$ in Lemmas 4.4, 4.5, and 4.6. The same argument as in the proof of Theorem 4.7 yields the desired result. ■

We now show that for any system $\mathcal{E}$ of equations, free set algebras that are free in $\mathsf{Wext}(\mathcal{E})$ relative to $\mathcal{E}$ are unique up to a bisimulation. This will follow as a corollary from the following result.

4.9 Theorem Let $\mathcal{F}$ and $\mathcal{G}$ be set algebras with supports F and G, respectively. Suppose there are set-maps $f : F \to G$ and $g : G \to F$; then $\mathcal{F}$ and $\mathcal{G}$ are bisimilar.

Proof. We need to find a bisimulation $\simeq$ with domain F and co-domain G. Obviously, for any F and G there is a bisimulation with domain $\subseteq F$ and co-domain $\subseteq G$: the least fixed point of $(\cdot)^{\forall\exists}$ will do. However, there is no guarantee that the least (or — for that matter — the greatest) fixed point will be *total* on F and G. To achieve that we need a bit more work.

Given two set maps f and g as in the statement of the theorem, define a relation R by putting

$$R(x, y) \iff (f(x) \equiv y) \vee (g(y) \equiv x).$$

Clearly, $\mathrm{dom}(R) = F$ and $\mathrm{rng}(R) = G$. We verify that $R \subseteq R^{\forall\exists}$. Assume $R(x, y)$ and $u \in x$; there are two cases according as $f(x) = y$ or $g(y) = x$.

1. If $f(x) = y$ then, since f is a set map, $f(u) \in f(x) = y$, and since $R(u, f(u))$ we have that there is $v \in y$ such that $R(u, v)$.
2. If $g(y) = x$ then $u \in g(y)$, and since g is a set map, $u \equiv g(v)$ for some $v \in y$, and clearly $R(u, v)$.

In either case, there is $v \in y$ such that $R(u, v)$. A symmetric argument shows that for all $v \in y$ there is $u \in x$ such that $R(u, v)$. This shows $R^{\forall\exists}(x, y)$, as desired.

By Remark 1.12, there is a bisimulation $\simeq$ extending R, and since $\mathrm{dom}(R) = F$ and $\mathrm{rng}(R) = G$, the same holds for $\simeq$. ■

4.10 Corollary Let $\mathcal{F}$ and $\mathcal{G}$ be free in $\mathsf{Wext}(\mathcal{E})$ relative to $\mathcal{E}$. Then $\mathcal{F}$ and $\mathcal{G}$ are bisimilar.

Proof. Since $\mathcal{F}$ is free, there is a set-map $f : F \to G$, and since $\mathcal{G}$ is free, there is a set map $g : G \to F$. ■

5 Further Properties

In this section, we briefly touch upon three topics. First we show that any set algebra in $K(\mathcal{E})$ generated by the 'unknowns' E is a set-image of any one of the free algebras in $K(\mathcal{E})$; next, we give an analogue of the classical projective properties for free algebras; and finally, we consider algebraic properties analogous to Hopficity investigate their relation to extensionality.

Part of the reason why free structures are important (in equational theories) is that for any given variety, any structure in that variety is the homomorphic image of a free structure. Thus, for instance, every group is the image of a free group, every abelian group is the image of a free abelian group, etc. This does not quite hold for set algebras in general, and the reason seems to be that set algebras are not an equational variety. However, we can obtain a first approximation to such a property: every set-algebra satisfying $\mathcal{E}$ via some map α and generated by $\mathrm{rng}(\alpha)$ is the homomorphic image of *any* free set algebra. Notice that the set algebra $\mathcal{S}[E]/\simeq_{\mathcal{E}}$ is generated by $\{[e] : e \in E\}$.

5.1 Theorem Let $K(\mathcal{E})$ be the class of all set algebras satisfying $\mathcal{E}$. Suppose $\mathcal{F}$ is free in $K(\mathcal{E})$, $\mathcal{S}$ is any structure in that class, and $\alpha : E \to \mathcal{S}$ a solution for $\mathcal{E}$. Then, if $\mathcal{S}$ is generated by $\alpha(E)$, there is a set-like homomorphism β from $\mathcal{F}$ *onto* $\mathcal{S}$.

Proof. With no loss in generality assume $E \subseteq \mathcal{F}$. Then, since $\mathcal{F}$ is free, we can extend α to a set-like homomorphism $\beta : \mathcal{F} \to \mathcal{S}$.

We need to see that β is surjective. Since $\mathcal{S}$ is generated by $\alpha(E)$, any $x \in \mathcal{F}$ has the form $t(\alpha(\overline{e}))$. Since β extends α, we have

$$t(\alpha(\overline{e})) = t(\beta(\overline{e})),$$

and since β is a homomorphism,

$$t(\beta(\overline{e})) = \beta(t(\overline{e})).$$

So β is onto. ■

Another classic property of free algebras is the *projective property*: if $\mathcal{F}$ is free, $\alpha : \mathcal{F} \to \mathcal{G}$ is a homomorphism, and $\beta : \mathcal{H} \to \mathcal{G}$ is an epimorphism, then there is a homomorphism $\gamma : \mathcal{F} \to \mathcal{H}$:

$$\begin{tikzcd} & \mathcal{H} \arrow[dr, "\beta"] & \\ \mathcal{F} \arrow[ur, "\gamma"] \arrow[rr, "\alpha"] & & \mathcal{G} \end{tikzcd}$$

In the case of set algebras we have a similar property, but it turns out that we can strengthen the consequent to γ being a set-map.

5.2 LEMMA Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be set-algebras such that $\mathcal{F}$ is free over E; assume that $\alpha : \mathcal{F} \to \mathcal{G}$ is a homomorphism, and $\beta : \mathcal{H} \to \mathcal{G}$ is an epimorphism. Then there is a set-like homomorphism $\gamma : \mathcal{F} \to \mathcal{H}$.

Proof. With no loss in generality assume $E \subseteq \mathcal{F}$. Then the identity over $\mathcal{F}$ is a solution for each equation $e \equiv t(\overline{e})$ in $\mathcal{E}$, i.e., $e = t(\overline{e})$. Then of course $\alpha(e) = \alpha(t(\overline{e}))$, and since α is a homomorphism, also $\alpha(e) = t(\alpha(\overline{e}))$. This shows that $\alpha \restriction E$ is a solution.

Now, since β is surjective, for each e in $\mathcal{F}$ find c in $\mathcal{H}$ such that $\alpha(e) = \beta(c)$. Then:

$$\beta(c) = \alpha(e) = t(\alpha(\overline{e})) = t(\beta(\overline{c})).$$

This shows that the map $e \mapsto \beta(c)$ is a solution in $\mathcal{H}$. By freeness, there is a set-map $\gamma : \mathcal{F} \to \mathcal{H}$.

■

It is well known that in certain varieties the projective property provides a characterization of the free structures. For instance, this is the case for (abelian) groups. This motivates the following conjecture.

5.3 CONJECTURE Suppose $\mathcal{F}$ is in $K(\mathcal{E})$. Let $\mathcal{F}$ have the property that whenever $\alpha : \mathcal{F} \to \mathcal{G}$ is a homomorphism, and $\beta : \mathcal{H} \to \mathcal{G}$ is an epimorphism then there is a set-like homomorphism $\gamma : \mathcal{F} \to \mathcal{H}$. Then $\mathcal{F}$ is free over E.

There is a property from universal algebra, known as *Hopficity* (see [8]), which in the case of set algebras turns out to have some connection with strong extensionality. In the following theorem we use the fact that if there is a strong homomorphism $h : \mathcal{S} \to \mathcal{S}'$ then $\mathcal{S}$ and $\mathcal{S}'$ satisfy exactly the same sentences not involving identity. (The result that follows has an analogue in which 'extensional' is replaced by 'weakly extensional' and '$=$' is replaced by '$\equiv$'. Since the argument is the same, we do not reproduce it.)

5.4 DEFINITION A set algebra $\mathcal{S}$ is *Hopfian* in a class K of structures if $\mathcal{S} \in K$, and every strong epimorphism $h : \mathcal{S} \to \mathcal{S}' \in K$ is injective.

5.5 THEOREM Every (strongly) extensional set algebra is Hopfian (in the class of all set algebras).

Proof. Let $\mathcal{S}$ be extensional, and $h : \mathcal{S} \to \mathcal{S}'$ a strong epimorphism. Suppose that $h(x) = h(y)$. Then obviously in $\mathcal{S}'$ it holds:

$$(\forall z \in \mathcal{S}')(z \in h(x) \leftrightarrow z \in h(y));$$

since h is onto, also

$$(\forall z \in \mathcal{S})(h(z) \in h(x) \leftrightarrow h(z) \in h(y));$$

since h is strong, in $\mathcal{S}$ it holds

$$(\forall z \in \mathcal{S})(z \in x \leftrightarrow z \in y);$$

Since $\mathcal{S}$ is extensional, $x = y$. So h is injective. ■

It is an interesting question whether all Hopfian set algebras are strongly extensional, or at least for which classes of set algebras Hopficity implies extensionality. We do not know the answer to these questions, but the following result appears to be in the neighborhood of what is wanted. By the way theorem 5.6 appears to establish the sort of interesting connection between algebraic and set-theoretic properties mentioned at the beginning.

5.6 THEOREM Suppose $\mathcal{S}$ is (set-theoretically) well-founded. Then the following are equivalent:

1. every set-map $h : \mathcal{S} \to \mathcal{S}'$ is injective;
2. $\mathcal{S}$ is extensional.

Proof. The direct implication (1) $\implies$ (2) does not require well-foundedness. Suppose extensionality fails in $\mathcal{S}$. So there are a and b such that $\forall x(x \in a \iff x \in b)$, and yet $a \neq b$. In particular, a is memberless if and only if b is memberless.

Let $\simeq$ be smallest relation such that $a \simeq b$ and closed under the clauses of reflexivity (ρ), symmetry (σ), transitivity (τ), and monotony (μ). One can show by induction that if $x \in y$ and $y \simeq z$ then z has members, and so is a set. Next, as in theorem 3.7, one shows that the quotient $\mathcal{S}/\simeq$ is a set algebra. Let $h : \mathcal{S} \to \mathcal{S}/\simeq$ be the natural homomorphism such that $h(x) = [x]$. Clearly h is not injective since $h(a) = h(b)$. It remains to show that h is a set map, which will follow when we establish that if $h(x) \in h(y)$ then $h(x) = h(z)$ for some $z \in y$. But this follows from the "converse of μ" (lemma 3.3): if $h(x) \in h(y)$ then $[x] \in [y]$. Since y has members, by the converse of μ we have that $x \simeq z$ for some $z \in y$, which gives $h(x) = h(z)$ for some $z \in y$, as required.

Now we take up the other implication, (2) $\implies$ (1). Suppose $\mathcal{S}$ is extensional well-founded and $h : \mathcal{S} \to \mathcal{S}'$ is a set map, in order to establish that h is injective. In particular, since $\mathcal{S}$ is well-founded, we can define the $\in$-rank $\mathsf{rk}(x)$ for each x in $\mathcal{S}$.

Suppose for contradiction that there are x and y such that $x \neq y$ but $h(x) = h(y)$, and suppose the ordinal $\mathsf{rk}(x) + \mathsf{rk}(y)$ is minimal among the pairs providing a counter-example to injectivity.

Now we argue as follows: suppose $z \in x$; then $h(z) \in h(x)$, and since $h(x) = h(y)$, also $h(z) \in h(y)$. Because of the back condition of set maps, there is a $z' \in y$ such that $h(z) = h(z')$. But $\mathsf{rk}(z) + \mathsf{rk}(z') < \mathsf{rk}(x) + \mathsf{rk}(y)$, so by minimality $z = z'$. It follows that $z \in y$. This shows that if $z \in x$ then $z \in y$. Reasoning symmetrically, we obtain also that if $z \in y$ then $z \in x$, and from these by extensionality we have $x = y$, contradicting the assumption. ■

We do not know whether the hypothesis of well-foundedness in theorem 5.6 is necessary. It appears to be so, but we do not have a counter-example to the theorem without it.

A Addendum

The condition of Theorem 5.6 is indeed necessary. Here are two counter-examples to the theorem.

A.1 EXAMPLE Let $\mathcal{M}$ be an *extensional* set algebra satisfying the following two additional conditions: $\mathcal{M}$ is *closed under complements* and contains a *universal set* V — i.e., the support of the set algebra exists as a set in $\mathcal{M}$. (We could, for instance, take $\mathcal{M}$ to be a model of Quine's "New Foundations".) Let $\simeq$ be the smallest (external) relation on V identifying the two sets V and

$V \setminus \{V\}$ and satisfying the clauses of reflexivity, symmetry, transitivity, and monotony as given in Definition 3.1. It is immediate to check that $\simeq$ is a fixed point of $(\cdot)^{\forall\exists}$, i.e., $\simeq$ is a bisimulation.[1] Let h be the canonical homomorphism of $\mathcal{M}$ over its quotient with respect to $\simeq$. Then it can be shown that h is indeed a set-map, which is, of course, non-injective.

The hypotheses on $\mathcal{M}$ in the previous example are quite strong. On closer scrutiny, all that is needed for the counter-example to go through is an extensional set algebra with two distinct bisimilar sets. Since one can show by $\in$-induction that any two bisimilar sets are identical, any such set set algebra needs to be non-well-founded. In the following example we show that such set algebras are a lot easier to come by than models of "New Foundations".

A.2 Example Let $\mathcal{M}$ be an extensional set algebra containing sets A_n and B_n for each $n \geq 0$, and such that:

$$\begin{aligned} &A_0 \neq B_0;\\ &A_{n+1} = \{A_n\}, \text{ for all } n;\\ &B_{n+1} = \{B_n\}, \text{ for all } n; \end{aligned}$$

(since $A_0 \neq B_0$, extensionality implies that $A_n \neq B_n$ for all n). A compactness argument suffices to obtain such a set algebra.

Now, let $\simeq$ be the smallest relation identifying A_n and B_n for all n, and closed under the usual clauses of reflexivity, symmetry, transistivity, and monotony. As before, $\simeq$ is a bisimulation, so that the canonical homomorphism h of $\mathcal{M}$ onto its quotient with respect to $\simeq$ is a non-injective set-map.

The above considerations suggest the following strengthening of the usual set-theoretic axiom of extensionality: say that $\mathcal{S}$ is *super-extensional* if and only if every set-map $h : \mathcal{S} \to \mathcal{S}'$ is injective. Clearly, extensionality and super-extensionality are equivalent for well-founded structures. It follows that super-extensionality is equivalent to the following condition: for every x, y in $\mathcal{S}$, if for some bisimulation R on $\mathcal{S}$ we have $R(x, y)$, then $x = y$.

B An Alternative presentation

We now briefly consider an alternative presentation of set algebras, closer to the approach of Joyal & Moerdijk [6]. Define a *set algebra* to be a tuple $\mathcal{S} = (M, \leq, +, s)$ where M, $+$ and s are as before, but $\leq$ is a binary relation corresponding to inclusion and satisfying:

Alt1 $x \leq x$ (reflexivity);

Alt2 if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity);

Alt3 $x \leq x + y$, and similarly $y \leq x + y$;

Alt4 if $x \leq z$ and $y \leq z$ then $x + y \leq z$;

Alt5 if $s(x) \leq y + z$ then $s(x) \leq y$ or $s(x) \leq z$ (irreducibility of successor);

Alt6 if $s(x) \leq s(y)$ then $s(y) \leq s(x)$;

---

[1] The observation that V and $V \setminus \{V\}$ are bisimilar is due to J. Barwise.

Alt7 if $s(x) \leq s(y)$ then $x \leq y$ (uniqueness of successor).

Then one could define membership by putting $x \in y$ iff $s(x) \leq y$, and a congruence $x \equiv y$ if and only if $x \leq y$ and $y \leq x$. One could then verify that the following all hold:

1. $x \in s(x)$;
2. if $x \in s(y)$ then $y \in s(x)$;
3. if $x \equiv y$ and $y \in z$ then $x \in z$;
4. $x \in y + z$ if and only if $x \in y$ or $x \in z$;
5. relation $\equiv$ is a congruence relation with respect to $\leq$ and $\in$.

This presentation is essentially equivalent to the one considered above.

References

[1] P. Aczel, **Non-Well-Founded Sets**, Center for the Study of Language and Information, Stanford, CA 1988.

[2] G. A. Antonelli, *Non-Well-Founded Sets via Revision Rules*, **Journal of Philosophical Logic** 23 (1994), pp. 633–79 (*Math. Rev.* 95k:03086).

[3] G. A. Antonelli, *Extensional Quotients for Type Theory and the Consistency Problem for NF*, **The Journal of Symbolic Logic**, v. 63 n. 1 (1998), pp. 247–261.

[4] J. Barwise & L. Moss, **Vicious Circles**, Center for the Study of Language and Information, Stanford, CA 1996.

[5] P. M. Cohn, **Universal Algebra**, D. Reidel Publishing Co., Dordrecht and Boston 1965 (1981).

[6] A. Joyal & I. Moerdijk, **Algebraic Set Theory**, London Mathematical Society lecture Note Series, n. 220, Cambridge University Press, Cambridge 1995.

[7] M. Mislove, L. Moss, & F. Oles, *Non-Well-Founded Sets Obtained from Ideal Fixed Points*, **Information and Computation** 93, n. 1 (1991) pp. 16–54.

[8] D. J. S. Robinson, **A Course in the Theory of Groups**, Springer Verlag, Berlin and New York, 1982.