FREGE ON IDENTITY STATEMENTS*

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1. Looking at the sweep of Frege's writings, one is struck by the coherence of his philosophical perspective; virtually all of the basic issues that animated his thinking are brought forth in his early work, and much of his subsequent thought can be seen as attempting to find the most cogent and coherent packaging to express these ideas.[1] One place where this can be seen most graphically is in Frege's known remarks on identity statements, a category into which he ultimately lumped mathematical equalities, containing the equals sign, with sentences of natural language like "Hesperus is Phosphorous" that contain a form of the verb "be."[2] With remarks spanning a period of 35 years from 1879 to 1914, Frege's perception of the importance of identity statements, especially for his views of logic and number, remained constant, stemming throughout from his understanding of the need for an identity symbol in his "conceptual notation" (*Begriffsschrift*) in order to obtain the generality required of a logistic system. But the presence of this symbol raised a problem that still perplexes us today.[3] On the one hand,

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*I am very pleased to be able to contribute this paper to a festschrift for Andrea Bonomi. This is not however, the paper I really wanted to write; I would have much rather have contributed a paper comparing the pianistic styles of Lennie Tristano and Bill Evans, which I think Andrea would have found much more fascinating than an essay devoted to an understanding of Frege's thinking. But I do not totally despair. Andrea's first paper published in English was entitled "On the Concept of Logical Form in Frege," so perhaps I can maintain some hope that this paper will appeal to lingering interests that Andrea wrote of in the past. I would like to thank Johannes Brandl, Ben Caplan, Bill Demopoulos, Bob Fiengo, Mark Kalderon, Patricia Marino, Gila Sher, Michael Thau, Dan Vest and especially Aldo Antonelli for very helpful discussion.

[1]In what follows, I will reference Frege's writings by name, with bibliographic listings given at the rear of the paper. References to other works are given in the notes. I will also for the most part utilize standard logical notations in lieu of Frege's.

[2]Included with the latter are statements with more prolix locutions for identity, such as "is the same as" or "is coincident with."

[3]See, for example, W.V.O Quine, *Philosophy of Logic* (Englewood Cliffs: Prentice-Hall, 1970), pp. 61 - 64.

identity statements play a logical role, licensing substitutivity; but yet they also express substantive propositions, to be proved or established. Is the identity symbol to be a logical or non-logical symbol? Both of Frege's approaches to identity statements, that of *Begriffsschrift* in 1879, and that of *Grundgesetze der Arithmetik* and "On Sense and Reference" in the early 1890's, although born from somewhat disparate considerations, bear strong similarities in the way they attempt to resolve this tension. Indeed, Frege toiled so diligently and profoundly at this resolution that it resulted in one of his most important and enduring contributions to philosophical thought, the distinction between sense and reference.

In *Begriffsschrift*, as the title implies, Frege's primary goal was to present a logical theory; part of the brilliance of this theory was in the way Frege saw how logic and semantics were related, and the importance of logical form for understanding the intimacy of this relation. Thus, Frege makes much of the logical form "*f*(*a*)," composed of a function-expression and an argument-expression, as transparently representing the application of function to object. Where this tight relation comes unstuck, as Frege saw it, was precisely at identity statements. From the outset of his explicitly logical explorations in *Begriffsschrift*, Frege thought it undeniable that among the basic symbols must be one for identity; otherwise, logic could not suffice as a general system of reasoning. To achieve this generality, Frege understood that not only must this symbol appear in propositions that can be true or false, it must also be a logical symbol; the truth of a statement of identity allows for a transition between propositions by substitution in the course of proof. Frege initially thought that he could connect the logical and semantic aspects of identity statements by a claim about the constituents of the logical form of identity statements, that they could be analyzed metalinguisticaly as relations between expressions. But by the time of *Grundlagen*, Frege had desisted from justifying an "identity of content" sign; with the emergence of the logicist project, considerations that did not weigh on Frege in *Begriffsschrift* made it clear that a different analysis was called for, for at the heart of this project were propositions that had to be construed as identity statements holding of logical objects, the most basic sorts of objects that logic concerns itself with, on Frege's view. That the analysis of the identity symbol as a sign of objectual identity was not a bar to identity statements playing their logical role was definitively clarified by Frege in *Grundgesetze*. But now the semantic issue of how identity statements could be truly, and not trivially, about something, loomed large, for something fundamental was presently at stake for Frege, the nature of number. In particular, Frege archly felt the need to meet and deflect the views and criticism leveled by the mathematical formalists, which of competing views most directly threatened Frege's foundational assumptions. Thus, from the 1890's on, the discussion shifts to the semantic analysis of identity statements, with center stage belonging to the justification of the central semantic relation, determination of reference, or as it is now called, *Sinn*.[4]

While Frege's initial analysis of identity statements was perhaps too heavily governed by views that seem anachronistic today, it was a well-considered view, as was his departure to a view intimate with the great later developments in his thinking. As noted, Frege strived to present a coherent philosophical picture; finding the right analysis of identity statements that fit within his philosophical and mathematical innovations, as well as his prejudices,[5] was essential to maintaining that coherence. Thus, to the extent that fundamental notions remained constant in his thought, we find similarity in his views of identity statements; to the extent that they evolve, discordance. In the course of the remarks to follow, I shall try to track the developments I have been describing so as to illuminate these similarities and discordances, with an eye to the central importance of identity statements in Frege's thinking. Our route will be roughly historical; after a brief discussion of rudimentary aspects of the system of *Begriffsschrift* (§2), we will turn to the identity of content analysis presented there (§§3-4). §§5 and 6 will explore the considerations that led Frege away from this account, to

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[4]Frege never explicitly remarked on the foundational implications for logic of assuming his more finely developed semantic views, never developing an intensional logic. This task was most famously undertaken by Alonzo Church; see his "A Formulation of the Logic of Sense and Denotation," in P. Henle, H. M. Kallen, and S. K. Langer, eds., *Structure, Method and Meaning: Essays in Honor of Henry M. Sheffer* (New York: The Liberal Arts Press, 1951), pp. 3 - 24 as well as subsequent papers revising the formulation.

[5]I have in mind here Frege's strong attachment, at least in his earlier work, to the central importance of the Kantian picture of the categories of judgements. For discussion of Kantian influences on Frege, see Philip Kitcher, "Frege's Epistemology," *Philosophical Review*, LXXXVIII, 2, (1979): 235 - 262.

the objectual identity account; §7, the criticism this account engendered from the mathematical formalists. Frege's answer to this challenge, via his theory of thought and judgement, is the topic of §8. The role of identity statements in the theory of number of *Grundgesetze* and Basic Law V is the concern of §9, and finally, in §10, we turn to Frege's ultimate meditation on identity statements, his most famous essay, "On Sense and Reference."

2. The task that Frege posed for himself in *Begriffsschrift* he sets out in the sub-title of the book; it is to present "a formula language, modeled upon that of arithmetic." The notation that Frege develops in *Begriffsschrift*, his "conceptual notation," carried through this project as a formalization and generalization of the way that mathematical reasoning, stripped of foolish assumptions and mistakes, is carried out. It constituted a substantial advance in logic for reasons that are well-known: for the first time it became possible to have a formal system in which rigorous, gapless proofs could be carried out. Frege's formalism for carrying through these proofs, although famously unwieldily, does have certain virtues absent from more familiar notations. In Frege's system, the formulas can be parsed into two fundamental parts, the *strokes* and the *symbols*. The strokes and symbols each have their own particular modes of combination. Basic for the joining of strokes are conditionalization and scope; for symbols, application of function to argument. These distinctions in junction, according to Frege, are themselves formal, or syntactic, but they map onto a distinction in content. The content of the strokes is given by their logical role, as specified by what amounts to a system of truth-tables; their significance lays in the relations holding among the (complete) symbols they connect together. The content of the symbols, on the other hand, is semantic, or as Frege would put it, "conceptual"; their significance resides in their designation of concepts and objects. Thus, the graphically distinct presentations of strokes and symbols mirrored distinct roles they play in the overall logistic system, thereby justifying the orthography of the system. Via the strokes, Frege characterized what we would call today the logical terms;

via the symbols, the non-logical.[6] Thus, we can rightly see Frege's primary concern in *Begriffsschrift* as being to give a theory of *logical form*, given with the appropriate formal rigor to support proof-theoretic certainty, conjoined with a semantic theory, which, given Frege's concerns, is presented at a more informal level.

The most basic sort of logical form in the *Begriffsschrift* system is that made up of the "content stroke" followed by a symbol for a content:

$$—— A$$

Frege says of such forms that he takes the content stroke "to mean that the content is unified" into a proposition; the "content stroke," Frege tells us, "serves also to relate any sign to the whole formed by the symbols that follow the stroke." (*Begriffsschrift*, p. 112.)[7] The content stroke is, if you will, the root stroke, to which others can be appended, so as to connect to other contents. Primary among the connected forms is that for the material conditional, from which, along with negation, Frege specifies the other truth-functional connectives. If a content, whatever the complexity of its connected parts, is prefaced by the content stroke, one further additional stroke may be added to the content stroke, the vertical "judgement stroke":[8]

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[6]To be clear, Frege did not have a *theory* of logical terms; he presented no necessary and sufficient conditions whereby we could classify terms as logical or non-logical. My point is just that his system embedded the distinction, and the notation graphically distinguished the terms.

[7]There are two complete translation of *Begriffsschrift* into English, one by Bynum, the other by Bauer-Mengelberg. There are also partial translations by Geach and by Beaney. Unless otherwise noted, quotations are from the Bynum translation.

[8]The system I am describing is that of *Begriffsschrift*. Frege adopts different terminology in *Grundgesetze*, re-labeling the "content-stroke" as the "horizontal." This makes clearer the purely notational role in the logic played by this stroke; it is there, so to speak, in order for other strokes to hang off of, but it has no interpretation as a logical term itself. Frege must have realized, given the redundancy of "— *A*" and "*A*," that his talk in *Begriffsschrift* of the content-stroke unifying a content added little in way of clarification, and even less so in light of the more sophisticated notion of content he presupposes in *Grundgesetze*, where he maintains that both of these denote (name) the same truth-value (§5). (In *Grundgesetze*, forms with the addition of the vertical line are assertions of truth-values, not judgements, in line with a changes in

$$\vdash\!\!\!\!\text{——}\ A.$$

The judgement stroke may be added just in case a content possesses the characteristic of being a possible *content of judgement*, or to use other terminology, it must be an *assertable content*.[9] We can proceed for instance from:

$$\text{——}\ f(a)$$

to:

$$\vdash\!\!\!\!\text{——}\ f(a),$$

where we have symbols for function and argument because what follows the content stroke qualifies as a possible content of judgement; we could not comparably proceed if only one or the other of these symbols occurred without the other. While Frege mentions a variety of reasons for the importance of distinguishing judged and unjudged propositions, the important point to recognize here is that in *Begriffsschrift*, *a judgement is a type of logical form*; (or to be more precise, there is an orthographically unique type of logical form that expresses a judgement).[10] Adding

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Frege's understanding of judgement; see discussion in §9 below.)

[9] Frege's term is *beurtheilbarer Inhalt*; see T. W. Bynum, "On the Life and Work of Gottlob Frege," in Gottlob Frege, *Conceptual Notation and Related Articles*, translated and edited by Terrell Ward Bynum, (Oxford: Clarendon Press, 1972), p. 79 - 80 for discussion of the various translations of this term.

[10] While Frege when justifying judgement often slips into a psychological idiom, I think it is unfair to accuse him, as Kenny does, of "a confusion between logic and what may broadly be called psychology." (Anthony Kenny, *Frege*, (Harmondsworth: Penguin Books, 1995), p. 36.) Kenny comes to this conclusion because he thinks that Frege in introducing the judgement symbol *defines* it in psychological terms. (p. 35) But this is to confuse defining and justifying - at no point does Frege define any logical symbol in "psychological" terms, although throughout his work he often justifies in such terms the need to define a symbol. That Frege would talk in this way is not too surprising, as the stated goal of *Begriffsschrift* is to model the fundamental properties of thought, "freeing thought from that which only the nature of the linguistic means of expression attaches to it," (p. 106) and one aspect of thought to be so liberated was judging. There are, as well, purely logical grounds justifying the judgement stroke, as Geach observes in his "Frege," in G.E.M Anscombe and P.T. Geach, *Three Philosophers*, (Ithaca: Cornell University Press, 1961), p. 133.

a vertical judgement stroke to a horizontal content stroke thus *transforms* one sort of logical form, representing a mere conceptual content, into another sort, representing a content of judgment. Or, as Frege puts it, the judgement stroke "converts the content of possible judgement into a judgement."[11]

Now Frege remarks with respect to contents of judgements that there may be various ways of parsing complex expressions as function and argument; for example, "Wittgenstein admired Frege" may be parsed as containing three different functions, depending upon whether "Frege,""Wittgenstein" or both denote arguments of the function. But whichever way the parse is made, "we can . . . apprehend the same conceptual content"; these differences in parse have "nothing to do with conceptual content, but only with our way of viewing it." (*Begriffsschrift*, p. 126.) Frege is insistent on this point as part of showing that there are parses as function and argument other than subject/predicate. But the remark is also revealing of Frege's view of the relation of form and content. Content is something that is *viewed* through form; it is through form that content is accessible to reasoning. It is because a content can be viewed that it can be judged. When so viewed it may be a content of a judgement, and the transition made to a logical form to which the judgement stroke has been added to the content stroke. If we ask why in a judgement

$$\vdash\!\!\!\!\!\text{———}\ A$$

is the conceptual content viewable, the answer is that there is a certain *conceptual analysis* phrased as function and argument that makes it so. A judgement is a *viewable* conceptual content.

3. In *Begriffsschrift*, immediately after introducing the logical strokes, Frege turns in §8 to the discussion of "identity of content," before his attention is drawn, in §9, to "The Function." It is telling that the discussion is juxtaposed in this way, given the janus-faced logical/semantic nature of statements of identity to which we have already alluded. Frege's notation at least makes it clear where he stands, for the identity symbol is not found among the strokes, but among the symbols, in with

[11]This elucidation is found in a footnote describing the symbols of the conceptual notation in "Boole's logical Calculus and the Concept-script," p. 11.

those things that have conceptual content. This placement immediately raises two issues:

> (i) What is the logical form of identity statements such that their content can be viewed?
> (ii) How do identity statements play their logical role, if the identity symbol is a "symbol," and hence non-logical?

To understand Frege's answers to these questions consider how he analyzes the logical form of identity statements in *Begriffsschrift*.

Frege opens the section entitled "Identity of Content" with the following remark: "Identity of content differs from conditionality and negation by relating to names, not to contents." (*Begriffsschrift*, p. 124.) He makes this more explicit, at the close of the section, as follows:[12]

> Now, let
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> $$\vdash (A \equiv B)$$
>
> mean: *the symbol A and the symbol B have the same conceptual content, so that A can always be replaced by B and conversely.*

The triple-bar "identity of content" symbol that we see in the judgement (type) displayed, which we may paraphrase as the relation "— has the same conceptual content as —," has the blanks filled in not with symbols for objects, but with symbols for symbols; judgements of such form are judgements of identity of content. From this definition, we can see right off that the answers to our questions will be interconnected in a very fundamental way, for what Frege maintains here is that identity statements can play the logical role they do just *because* the conceptual content of the symbols to be substituted one for another is the same.[13]

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[12]The italicized clause is from the Geach translation, p. 12.

[13]There is an exegetical issue here. Does Frege intend the substitution clause to be part of the "meaning" of judgements of identity of content, or does he take it to be a parenthetical remark about the logical role of judgements of identity of content within

Early on in *Begriffsschrift* Frege remarks that "the only thing considered in a judgement is that which influences its *possible consequences*. Everything necessary for a correct inference is fully expressed; but what is not necessary usually is not indicated" (*Begriffsschrift*, p. 113.) Given this, it must have been that Frege thought that judgements of identity of content expressed the information necessary in order for inference to proceed properly, and moreover that this information must be metalinguistic. Why would he have thought that? One of the things upon which Frege was most insistent was the *formal* nature of proof in a logistic system. What Frege meant by this is that while inferential relations hold between thoughts, we can only determine whether one thought follows from another with respect to the *form* by which that thought is expressed. Proofs proceed, if you will, in terms of what expressions (simple or complex) look like. Thus, because of the nature of the notational system, proofs could be carried out by proceeding from step to step via formal matching of symbols; *how* one can proceed in a proof in virtue of such matching is stipulated by rules of inference. Usually, these stipulations are made externally to the system; think of *modus ponens*, which gives license to detach the consequent from the antecedent of a conditional. But what if a pattern of inference were licensed by some sort of proposition that occurs *internally* to the system; what sort of information would it need to carry? Presumably, it too would have to carry the pertinent formal information about the symbols involved needed to proceed from one step in a proof to another; that is, it would have to carry metalinguistic information. In the case of substitution, if it is to be licensed by a statement of identity, then that statement must carry the pertinent information about the *symbols* such that they can be substituted one for another. If we have a judgement that says of symbols that they have the

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the overall logical system? While the former is perhaps indicated by Frege's typographical use of italics, the latter interpretation is perhaps more plausible, given the redundancy that would be introduced between the definition and the axiomatic governance of substitution that Frege specifies with proposition (52) of *Begriffsschrift*. Regardless, what is important here is the apparent causal connection Frege sees between identity of content and substitution; one substantial change we find in *Grundgesetze*, where Frege assumes objectual identity, is that this connection is broken. Nothing about an identity statement, in and of itself, implies anything about substitution of symbols.

same content, e.g. "$c \equiv d$," then we can move in a proof from the judgement:

$$\vdash P(c)$$

to the judgement:

$$\vdash P(d).$$

Frege characterizes such inferences by proposition (52):

$$\vdash (c \equiv d) \supset (f(c) \supset f(d)),$$

glossed with the remark that it "says that we may replace *c* everywhere by *d*, if $c \equiv d$" (*Begriffsschrift*, p. 162).

If capturing information relevant for inference is indeed the reason for the metalinguistic identity of content, then the following thought occurs: couldn't this information be exported and stated as a rule of inference? Couldn't we just transpose the basic proposition into a rule of inference in an informationally neutral way? According to the following remark, from "Boole's logical Calculus and the Concept-script," dated 1880/81,[14] Frege contemplated doing just this not long after finishing *Begriffsschrift*:[15]

> In the preface of my *Begriffsschrift* I already said that the restriction to single rule of inference which I there laid down was to be dropped in later developments. This is achieved by converting what was expressed as a judgement in a formula into a rule of inference. I do this with formulae (52) and (53) [i.e. the law of self-identity] of my *Begriffsschrift*, whose content I render by the rule: in any judgement you may replace one symbol by another, if you add as a condition the equation between the two.

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[14]Although submitted to three different journals, the paper never saw print; see Bynum's remark on p. 21 of "On the Life and Work of Gottlob Frege."

[15]"Boole's logical Calculus, p. 29.

Observe that Frege describes the proposed rule of inference in words cast in a manner very similar to those used to describe basic proposition (52); he speaks here as well metalinguistically of the relation of symbols. So it seems that as long as the information relevant to proof remains constant, basic propositions can be transformed into rules of inference; these are two ways of saying the same thing, one way internal, the other external, to the system.

While Frege's remarks illuminate how he understood the logical aspects of judgements of identity of content, he unfortunately is not entirely clear in elucidating what advantage he saw in stating substitution under identity (of content) as a rule of inference. We may, however, conjecture along the following lines. Stating substitution under identity as a basic proposition internal to the system forced upon Frege a bifurcation of symbols. Although the symbols that occur in the judgements that are inferentially related are, as Frege puts it, "representatives of their contents," what are substituted in the conceptual notation are symbols, so we must be able to recognize that it is the symbol "*b*" that is being substituted for the symbol "*a*." "Thus," Frege says, "with the introduction of a symbol for identity of content, a bifurcation is necessarily introduced into the meaning of every symbol, the same symbols standing at times for their contents, at times for themselves." (*Begriffsschrift*, p. 124). If our concern is just with the thoughts that partake of the inference, then the symbols stand for their content; but if our concern is with the proof of that inference, then we must be able to see the symbols as standing for themselves. Symbols shimmer between these two ways of being seen. Note, however, that Frege apparently did not feel any compunction to adopt such a bifurcation by assuming *modus ponens* as a rule of inference, and for good reason. In a proof, we proceed from step to step by recognizing that there is formal matching of symbols, and in order to state which matchings are legitimate we need symbols that stand for symbols (cf. the contemporary usage of schematic letters and corner quotes). The symbols that are so recognized, however, are those that are *used* in the statements that are inferentially related; mention of the symbols to be recognized in statements that may be inferentially related does not require that *in* the forms so related that the symbols themselves be *mentioned*. In the case of *modus ponens*, validating that "*b*" follows from "*a*" and "$a \supset b$" does not require that any of the symbols be mentioned *in the object language*. Thus, while stating *modus ponens* has a metalinguistic character, *qua* rule of inference, it would only be to confuse use and mention to bifurcate the symbols of the language. In a comparable way, there would be no call for bifurcation from substitution if it were characterized as a rule of inference, so that the artifice of taking the symbols of the conceptual notation as standing for anything other than their contents could be abandoned, for as Frege points out, it is only judgements of identity of content that require this. No longer would symbols need to stand both for their contents and themselves, a considerable simplification of the conceptual notation. But notice that if symbols are no longer bifurcated, then there is no longer any place for an identity of content symbol; identity will have to be otherwise defined. Frege recognized this in comparing his system to Boole's: "The first thing one notices is that Boole uses a greater number of signs. Indeed I too have an identity sign, but I use it between contents of possible judgement almost exclusively to stipulate the sense of a new designation. Furthermore I now no longer regard it as a primitive sign but would define it by means of others."("Boole's logical Calculus," p. 35-6.) Although he says nothing more on the matter, his making it a virtue that his system has fewer signs than Boole's indicates that Frege's initial rationale for moving away from the *Begriffsschrift* theory was that it simplified the conceptual notation. But aside from this, Frege most likely would not have seen any problem in returning to the prior approach; however, considerations were to shortly weigh in, starting with *Grundlagen*, that would move Frege to a rather different view.[16]

Returning to the *Begriffsschrift* account, we can place our finger on the reason that Frege adopted the metalinguistic slant of judgements of identity of content in that he thought that in this way the symbolization of identity could play its logical role, allowing proper movement from step to step in proofs. We must be able to see in a

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[16]Notice that if our conjecture is along the right lines, Frege would not have seen a use/mention *confusion* in the *Begriffsschrift* theory at this point in his thinking, although he may have come to see it that way later on. Church, for one, thought so: "If use and mention are not to be confused, the idea of identity as a relation between names renders a formal treatment of the logic of identity all but impossible. Solution of this difficulty is made the central theme of *Über sinn und Bedeutung* and is actually a prerequisite to Frege's treatment of identity in *Grundgesetze der Arithmetik*." (*ibid*, p. 3)

judgement of identity that *symbols* have identical contents, for otherwise such judgements would not fully express "everything necessary for a correct inference." Although there is nothing in principle barring the introduction of a symbol for objectual identity into the conceptual notation, it should be clear, given Frege's logical literalism in *Begriffsschrift,* that statements of objectual identity would not suffice for the job at hand, for they would not contain all the needed information. All that would be expressed in such a judgement would be that there is a unitary content, and this would at best beg the question of how substitution of an object for itself could lead from one distinct form to another. (Since the terms of an objectual identity are just "representatives of their contents," there is one and only one content in question.) Insofar as Frege would have seen a problem with statements of objectual identity in *Begriffsschrift* it was that they would not provide the right information for inference, and would not play the logical role Frege demanded of them. Note that this conclusion arises orthogonally to the semantic relation of "$a = b$" to "$a = a$." In fact, Frege nowhere in *Begriffsschrift* mentions the semantical problem that has come to be known as "Frege's Puzzle"- that "$a = a$" and true "$a = b$" express the same proposition (have the same content) - and there seems to be no reason that he would, for the problem for Frege would have been with "$a = b$" itself, and that problem would be a logical problem.[17]

4. In §8 of *Begriffsschrift*, after informally introducing the notion of identity of content, Frege turns immediately to a possible objection. We

[17] What awareness Frege had of the issue put this way at the time of writing *Begriffsschrift* is open to debate. Hans Sluga, in *Gottlob Frege* (London: Routledge and Kegan Paul, 1980) and "Semantic Content and Cognitive Sense" in L. Haaparanta and J. Hintikka, eds., *Frege Synthesized: Essays on the Philosophical and Foundational Work of Gottlob Frege* (Dordrecht: D. Reidel, 1986), points out that there is recognition of the problem, in the context of Kantian views, by Lotze in his *Logic* of 1874, and that he developed a position not unlike Frege's. This is one of the bases for Sluga's argument that Lotze was a significant influence on Frege, a view that has been extensively challenged by Michael Dumment, in *The Interpretation of Frege's Philosophy* (Cambridge: Harvard University Press, 1981) and in a number of articles collected in *Frege and Other Philosophers* (Oxford: Clarendon Press, 1991). Dummett does make a case that Frege had read Lotze *prior* to 1879, but believes him not to have held Lotze in particularly high esteem, never mentioning him by name in his writings. But regardless, it is clear that the significance of the problem did not come home to roost for Frege until much later, and then arising from a rather different source. See discussion in §7, especially footnote 43.

may put it as follows. Why, we ask, if "*A*" and "*B*" have the same content, do we need a symbol for identity of content? An initial answer is that in order to connect the expressions so as to form a judgement that *says* they have the same content we must recognize the difference between there being identity of content and *expressing* identity of content in the conceptual notation; we want to be able to express identity of content because this is what warrants substitutions of formally unlike symbols. This may appear, however, a rather hollow justification for if the reason for having an identity symbol is just that we have unlike symbols with the same content, doesn't this just show bad design of the system?[18] Isn't the identity symbol just an artefact, needed only because of an assumption about symbols, and eliminable by requiring a biunique relation between symbols and content? Yes, Frege tells us, but *only if* "it were here a matter of something pertaining only to *expression*, not to *thought*."[19] That is, the criticism would be appropriate if a judgement of identity of content was no more than an assertion of coreference. But this Frege denies holds of his theory; in it the matter rather pertains *both* to expressions *and* to thought, and it is this that ultimately justifies having an identity symbol within the conceptual notation.

Frege puts matters in the following way:

> . . . the same content can be fully determined in different ways; but, that the *same content*, in a particular case, is actually given by *two different modes of determination* is the content of a *judgement*. Before this can be made, we must supply two different names, corresponding to the two [different] modes of determination for the thing thus determined. But the judgement requires for its expression a symbol for identity of content to combine the two names. It follows from this that different names for the same content are not always merely an indifferent matter of form; but rather, if they are associated with different

[18]Wittgenstein thought so; see §5.53*ff* of the *Tractatus*.

[19]I find that the translation by Geach of this passage from *Begriffsschrift* (p. 11) reads better than Bynum's: "what we are dealing with pertains merely to the *expression*, and *not to the thought*." (*Begriffsschrift*, p. 124.)

> modes of determination, they concern the very heart of the matter.[20]

The question facing Frege is why would we want to have unlike expressions of the same content; what *justifies* this sort of multiplicity of symbols? It is justified, Frege tells us, because expressions may be related to contents in more than one way. The relation of expressions and content is not simplex or direct, but rather is mediated, by a mode of determination (*Bestimmungsweise*). Contents may be "given" in more than one way, and if anything is biunique, it is the relation between expressions and modes of determination of their content. Now we need to keep things straight about this sort of relation in a way that can be symbolically encoded in the conceptual notation; we do this by associating different labels - symbols - with distinct modes of determination. But then all we can conclude from the occurrence of distinct symbols is that they have distinct modes of determination, but *not* that they have distinct contents. Whether they have the same or different content is left open by the notation. It is closed by a judgement of identity of content; i.e. a logical form in which the symbols fall on either side of the identity of content sign. If there are two different symbols in such a judgement, it follows that the content is given in that judgement by two different modes of determination, (for otherwise there would not be distinct symbols).[21]

Modes of determination, Frege thus tells us, are what justify

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[20]*Begriffsschrift,* pp. 125 - 6.

[21]On Frege's way of putting things in *Begriffsschrift*, it is only assumed that *if* there are distinct modes of determination, they will be associated with distinct symbols; what is not clear is whether Frege allows a *single* mode to be associated with distinct symbols. If this were allowed, however, there would not be a need for a judgement of identity of content, for that would be already determined by the very fact that they are associated with the same mode of determination. The issue would then only be one of notation. Suppose that the mode of determination associated with "3" and that associated with "III" are the same; it would then be trivial to judge identity of content, just as it would be to judge identity of content of repetitions of "3." Thus, this case is irrelevant to the matter at hand, although it may not be irrelevant to say that they are *translations*; i.e. that "3" and "III" are the symbols in distinct notational systems that are associated with the mode of determination in question. (This is in essence the argument Frege makes in "On Sense and Reference," in which he explicitly allows distinct symbols to have the same sense.)

judgements of identity of content; but more than that, it is through this justification that we can touch the thought, for this justification will provide information fixing the judgements within the *categories* of thought. In the *Grundlagen*, Frege asks, "Whence do we derive the justification for [a content's] assertion?" He answers as follows:[22]

> Now these distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgement but the justification for making the judgement. Where there is no such justification for making the judgement, the possibility of drawing the distinction vanishes.

The justification to which Frege refers, we must understand, is not itself part of the content of the judgement per se, and is thus not represented in the judgement; it is rather *extra*-notational. Modes of determination are thus in no way *represented* in judgement above and beyond that which follows about them in virtue of the occurrence of (distinct) symbols. What is *expressed* by a judgement of identity of content via its representation in the conceptual notation is solely that the symbols have identical content; the justification for such a judgement, and hence its connection to "thought," is not so represented in such logical forms. This is not to say, however, that modes of determination cannot themselves be expressed as contents of judgements, and in fact that they can be is of importance in determining the category of judgements of identity of content.

Frege remarks that "If in carrying out [the proof of a proposition], we come only on general logical laws and on definitions, then the truth is an analytic one. . . . If, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one."(*Grundlagen*, p. 4.) Thus, if we are to categorize a judgement we must see what sort of premisses would justify its truth; what sort of premisses would be needed for its proof. In *Begriffsschrift*, in way of illustration, Frege asks us to consider two points on a circle, "*A*" and "*B*," where *A* is a fixed point on the

[22] *Grundlagen*, p. 3

circumference, “given through perception,” and *B* is computed as the (rotating) point of intersection of a line from *A* to the circumference. Frege then asks “What point corresponds to the position of the straight line when it is perpendicular to the diameter?” The answer is *A*; *A* and *B* are the same point. “Thus, in this case,” Frege now tells us, “the name *B* has the same content as the name *A*; and yet we could not have used only one name from the beginning since the justification for doing so is first given by our answer. The same point is determined in two ways”. (*Begriffsschrift*, p. 125.) What Frege has illustrated here is a “proof” of the judgement that “*A*” and “*B*” have the same content; the role of the modes of determination is that they stand as premisses of this proof. These modes of determination, however, are synthetic, one being geometrical, the other perceptual. Thus, because of the way in which this conclusion was reached, in this case, Frege puts it, “the judgement as to identity of content is, in Kant’s sense, synthetic.”

Judged by conceptual content, introducing a new atomic symbol into the conceptual notation with the same content as some other atomic symbol, does not, in and of itself, change the expressiveness of the system, unless by that introduction there is some judgement that can be expressed that otherwise could not be. This obtains in the case at hand, for a synthetic judgement has become possible, established by a proof of that judgement, where the critical premisses in the proof that fix its status are the modes of determination. Without such modes of determination there would be no way to establish that distinct expressions, atomic or not, play distinct roles in proofs, even though they have the same content.[23]

Before continuing, let us quickly survey the territory we have covered. In order to have a logistic system sufficiently general so as to serve as a general system of reasoning, Frege needed a symbol for an identity relation in the conceptual notation. Frege’s initial thought was that in order for it to play its logical role, it had to be identity of content; i.e. metalinguistic. This does not, however, detach judgements of identity from thought, which Frege took to mean from the Kantian categories of thought. One might think this because such judgements are ostensibly about coreference of expressions. But they are about something more, for the expressions related are associated with modes

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[23]My thinking in this section has been influenced by remarks of Mark Kalderon.

of determination, and from these we can deduce the connection to thought. In 1879, we thus see two themes in Frege's thinking that strike us as perhaps somewhat antiquated. One is his logical literalism, from which the need to mention the terms of the identity in order to explicate their logical role follows. The other is his adherence to the enduring influence of fundamental Kantian notions. In contrast, his concern about identity is strikingly modern, for in *Begriffsschrift* we see clearly illuminated for the first time the tension associated with identity statements, between their logical role and their status as expressing contentful propositions, which may be true of false.

5. In introducing modes of determination, Frege's concern was with how to justify having a symbol of identity of content within the conceptual notation in the face of an objection that it would be otiose, for we could just forbid in the first place symbols with the same conceptual content. But such an objection, at least under one interpretation, is patently absurd, if it is to mean that there could not be both a simple and a complex term (or multiple complex terms) for the same content, that we could not, for instance, name and describe things. Rather, the objection is only sensible if the issue is whether there can be more than one simple *atomic* term for any given content. Given his goal of having a generally applicable system for reasoning regardless of the content of propositions, Frege must allow for this, as we commonly come across in the sciences more than one simple term for a single thing. Showing by empirical proof, (i.e. proof from empirical assumptions), that atomic terms apply to the same thing can be the essence of a scientific discovery. If it is justified to have more than one atomic term for a given object, we will then need an identity of content sign, if we are to allow for the assertion that they are terms for the same thing. Such assertions are not just about the expressions, but also pertain to the thought because each atomic term is itself justified by being associated with distinct modes of determination. Such distinct modes of determination thus give good reason for multiple atomic terms for a given object.

Seen in this light, Frege's choice of a geometric example to illustrate becomes understandable, although at first it might seem odd. For although the system of *Begriffsschrift* is intended as a general system, applicable to any domain of reasoning, Frege's particular concern is with the nature of arithmetical judgements; the latter part of the book establishes notions, in particular that of ancestral of a relation, that were to play a vital role in Frege's forthcoming logicism. Yet Frege chooses not an arithmetical example, but a geometrical one, and one might be curious why this is. The reason he avoids an arithmetical example is that arithmetic is one domain where we get along with a primitive biunique symbol/content relation - there is one and only one atomic symbol for each number, its numeral, and hence there are no simple expressions of which we need to say that they have the same content.[24] Thus, one way of seeing the problem with arithmetic is that it just doesn't have the right level of generality to serve the point Frege wishes to make; the trivial "2 ≡ 2" wouldn't do the trick. But with regard to this aspect of arithmetic, there is also a more subtle issue at play, which, when understood, sheds light on the path Frege took to his "mature" view of identity, that of *Grundgesetze* and "On Sense and Reference."

There is a curious aspect of Frege's presentation in *Begriffsschrift* and in the articles he wrote contemporaneous with it in explication of his system. While Frege introduced the identity of content sign in order to insure the general applicability of the conceptual notation to all domains of inquiry, there is one crucial place where he does not employ this symbol. This is in judgements of arithmetic equality. In stating sums, for instance, Frege nowhere writes them as judgements of identity of content; he never writes an equation as "2+3 ≡ 5," but rather as"2+3 = 5." Although it is not altogether apparent in *Begriffsschrift* proper,[25] he makes this usage clear in a lecture entitled "Applications of the 'Conceptual Notation'," presented in late January 1879, just six weeks after Frege completed *Begriffsschrift*, and in "Boole's logical Calculus and the Concept-script," composed in 1880.

[24]A number may be designated by more than one numeral if they are in different bases. But this would be a case of how a number is designated in, so to speak, different *languages*, and would be an issue of translation, rightly a metalinguistic matter. Equating numerals in different bases would be no different than equating Arabic and Roman numerals; it would be to say no more than that "3" and "III" are different ways of representing or *inscribing* the same numeral (number-name). This relation, however, is not identity of content as Frege saw matters, for it does not correspond to a difference in mode of determination; the number 3 - that object - is determined in the same way regardless of whether we write its numeral "3," "III," or "three."

[25]One place where Frege gives a judgement with equality is when introducing the material conditional in §5 - he uses "3×7 = 21" as an example.

In both of these essays, Frege sought to show how the conceptual notation could be employed to express certain complex arithmetic propositions, providing in course a range of illustrations. To take but one example, in "Applications" in discussing the theorem that every positive whole number can be "represented" as the sum of four squares, Frege considers the following equation:

$$30 = a^2 + d^2 + e^2 + g^2.$$

Thus, the practice we observe is that Frege uniformly employed an *equality* symbol, symbolized as "=", in arithmetic propositions; this is in addition to his assumption of the identity of content symbol ("≡").

Given that Frege employed two different symbols, it is reasonable to assume that he thought that they had two different meanings; it would not have been Frege's style to have two distinct symbols with the very same meaning. However, of these two symbols, Frege only specifies the meaning of one of them, the identity of content sign; no specification is given of the equality sign. But why should there have been? Frege would have no more have perceived a need to define the arithmetic equality sign as he would have thought it necessary to define the plus-sign. He would have thought the meaning to be obvious; in accordance with mathematical practice,[26] the intuitive equation of an arithmetical operation with either some number, or with some distinct arithmetical operation. The equality relation as we find it in "2+4 = 6" equates the sum-operation applied to 2 and 4 with the number 6; in "2+4 = 2×3" with the product-operation applied to 2 and 3. These are judgements, as Frege puts it, "which treat of the equality of numbers which have been generated in different ways." ("Boole's logical Calculus," p. 13)

The equality symbol, as Frege would have construed it at this time, is thus part of the vocabulary of arithmetic, and just as other areas of mathematics may also sport their own specific equivalence relations, e.g. congruence in geometry, equality represents an explicitly arithmetic concept. In introducing the identity of content sign as the symbol of identity in *Begriffsschrift,* Frege viewed it as a novel symbol within the

[26]Including Frege's own; see his usage in the mathematical work he had undertaken by the time of *Begriffsschrift,* in particular "Methods of Calculation" (1874).

conceptual notation, not bound to any specific domain, and not as a replacement for a well-known sign of arithmetic. Indeed, as far as arithmetic is concerned, equality and identity are complementary notions; those judgements containing the equality sign could not be replaced by judgements with the identity of content sign standing in its stead, nor vice versa, with identity of content replaced by equality. Unlike identity of content, equality is not a metalinguistic relation; the symbols that occur in judgements of equality stand for their contents. Thus, in holding between the sum operation as applied to the numbers 2 and 4 on the one hand, and the number 6 on the other, equality holds between distinct conceptual contents. But then we could not replace arithmetical equality with identity of content. For "2+4 ≡ 6" to be true, "2+4" and "6" must have the *same* conceptual content; if "2+4 = 6" is true, "2+4 ≡ 6" will be false. Equality, in what Frege would have taken as its common arithmetic sense, although an equivalence relation, is not identity.[27]

Notice that aside from not being metalinguistic, judgements of equality differ from judgements of identity of content in another important way. It is characteristic of the latter judgements that significant information relevant to conceptual content is secreted away, revealed as modes of determination. But with equalities, the mathematically relevant information is displayed in plain sight; the mathematically relevant information expressed by "2+4 = 6" is just *that* the sum of 2 and 4 is 6. There is no need of modes of determination to reveal this information; once we have explicated the conceptual content of the judgement, we will have elucidated all the relevant information that it expresses. The circumstances with judgements of equality are no different in this regard than with a judgement such as "John left." Where modes of determination are called for is just where there is a relation that does not reveal the relevant information, in particular where we say of two atomic names that they have the same conceptual content. But, as noted, this is not a circumstance that we find in arithmetic, given that there is only one numeral per number, and hence for which there would be no place for judgements of identity of content.

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[27]Bear in mind that in *Begriffsschrift* Frege did not have a generalized notion of denotation, such that "2+4" would denote its value, i.e. the number 6. This was only fully realized considerably later, at the time of *Grundgesetze*.

If these remarks are along the right track, it indicates that Frege saw the extent of judgements of identity of content as quite circumscribed. The operant notion here is *judgement*, for although in *Begriffsschrift* Frege never clearly specifies how identity of content is to be deployed, he does distinguish the *judgement* (assertion) of identity of content, and the *stipulation* of such identity. The primary case that Frege remarks upon of the latter use is definition.[28] Distinguishing these cases, Frege remarks that a definition "does not say "The right side of the equation has the same content as the left side"" - this is what a judgement would say - "but, "They are to have the same content." [A definition] is therefore not a judgement . . .The only aim of such definitions is to bring about an extrinsic simplification by the establishment of an abbreviation" (*Begriffsschrift*, p. 167-8), so as to formally simplify proofs and make them more comprehensible. There are two characteristics of this stipulative use for definition to remark upon. First, while definitions are not judgements, and are distinctly indicated in the conceptual notation by the use of a special symbol, once the definition is laid down it may be employed as a judgement, and play the logical role of such judgements:[29]

> Although originally [a definition] is not a judgement, still it is readily converted into one; for once the meaning of the new symbols is specified, it remains fixed from then on; and therefore [a definition] holds also as a judgement, but as an analytic one, since we can only get out what was put into the new symbols [in the first place].

A definition *qua* judgement is analytic because it is trivial, being merely abbreviatory; there are no distinct modes of presentation to call upon for the terms of the judgement, for we *would* then be getting more out of the

[28]Frege also uses identity of content to form statements that express the stipulation of specific values for variables in a proof; cf. his usage in the derivation in *Begriffsschrift* of proposition (68), in which he gives "$(\forall \mathfrak{a}\ f(\mathfrak{a})) \equiv b$" to state the value of a propositional variable.

[29]*Begriffsschrift*, p. 168. The final parenthetical emendation of the quotation appears in the quoted text, placed there by the translator.

new symbols than we put in. Second, in the statement of a definition there will always be a complex expression, the one that is being abbreviated. In contrast, Frege’s practice indicates his intention that we take his use of the singular definite article seriously when he says in characterizing non-stipulative judgements of identity of content, those to which modes of presentation are pertinent, that “*das Zeichen A und das Zeichen B*” are related by identity of content. Apparently only single, atomic symbols can stand to the sides of the identity of content sign.

Notice that nothing in what we have said proscribes “$2+4 \equiv 6$”; it is just that it would only be construed as *defining* the numeral “6” by stipulating that it has the same conceptual content. as “2+4.” There is one place in arithmetic, however, where we could imaginably employ identity of content; observe that not only is “$\neg (2 = 3)$” true, so too is “$\neg (2 \equiv 3)$.” Frege actual usage is to employ the equality sign. In “Applications,” when characterizing prime numbers, Frege gives the clause “$\neg (\mathfrak{d} = a)$” as part of expressing that for any number *a*, it is indivisible by any positive whole number $\mathfrak{d}$ in the sequence beginning with 2, *such that* $\mathfrak{d}$ *is different from a*.[30] (On Frege’s use of variables, roman characters are bound by an implicit maximally wide scope universal quantifier, while gothic characters are bound by narrower scope quantifiers that are explicitly indicated.) This bolsters our initial conclusion, that Frege only employed judgements of identity of content

[30]At least this is how it is rendered in the German original; in the English translation by Bynum it is rendered with identity of content: “$\neg (\mathfrak{d} \equiv a)$”. If Bynum has correctly brought the text into line with Frege’s intended usage, then it indicates that Frege would have held a stronger thesis, that the identity of content sign would have to occur between simple atomic symbols, presumably as a stipulation of the syntax of the conceptual notation. I have deferred, however, to the usage in the original; that it is the intended usage is supported by the formula Frege gives in “Boole’s logical Calculus” (p. 23) to express that 13 is prime, which is identical to the formula given in “Applications,” save appropriate substitutions. It is given with the equality, not the identity of content, sign. In conversations with Prof. Bynum, he was unfortunately unable to recollect why he had made the noted change from the German original, aside from observing that it would have been reasonable to assume that so shortly after finishing *Begriffsschrift*, in a lecture devoted to explicating the system of that book, that Frege would have employed the full range of notions introduced there. This assumes, of course, that Frege inadvertently or mistakenly used the equality sign in this particular place, as opposed to his proper usage of this sign in other places, such as stating sums. I would like to thank Ignacio Angelelli and Christian Thiel, as well as Terry Bynum, for discussion of this point.

where modes of determination are involved. Since we are dealing with *non*-identities, we do not have two names for the same number, but rather names of different numbers, and hence there is no call for modes of determination. "$\neg$ (2 = 3)" by itself is sufficient to express that the two numbers 2 and 3 are distinct.[31]

At the time of *Begriffsschrift*, it appears clear that Frege does not fold mathematical equality into identity of content, and that he had two symbols, one arithmetic, the other, while not outright barred from arithmetic, since it is general to the logical of all domains, of no utility there. But Frege no doubt would have become aware of a substantial, and glaring, logical problem in this view. The problem is that in *Begriffsschrift*, substitution is characterized only under identity of content, by proposition (52):

$$\vdash (c \equiv d) \supset (f(c) \supset f(d)).$$

What is *not* to be found in *Begriffsschrift* is a characterization of substitution under equality; equations do not fall under the proposition above. Although Frege may have thought that since equality is an arithmetic notion, the substitution of equals for equals would be specified within arithmetic,[32] and not by a basic law of logic, it is still hard to imagine that it would not have irked him that substitution was characterized distinctly with respect to the two notions, but in ways that would be completely parallel logically. This redundancy would surely have indicated that something was amiss. The natural response, of course, would be to

[31] In "Applications," Frege gives the following formula, where "$\cong$"stands for congruence:

$$\forall U\ (CD \cong CU \supset (BD \cong BU \supset D \equiv U)).$$

"This is the case," Frege says, "when and only when *D* lies on the straight line determined by *B* and *C*." (p. 204) Unlike in arithmetic, however, Frege has no other option but to use the identity of content sign in this formula, since points are not the sort of things that are congruent. (Congruence, as Frege specifies, holds between two pairs of points.) Recall that Frege established by example in *Begriffsschrift* that points may be multiply designated, with each designation being associated with a different mode of determination.

[32] In fact, Frege addressed the issue of how to derive arithmetic formulas from one another via substitution in work prior to *Begriffsschrift*, specifically in *Methods of Calculation*, his thesis of 1874; cf. pp. 61 - 64.

seek a unification, one general notion applicable in all cases. This is precisely what Frege does, beginning with *Grundlagen*, in 1884.

6. Given that we have two overlapping notions, an initial strategy in seeking to unify them would be to see if one could be reduced to the other. This is not, however, the strategy that Frege takes; rather he opts for a replacement strategy, introducing a new notion that subsumes the old ones. This notion is objectual identity; any statement that validates substitution is now to be analyzed as an *identity statement*. This includes not only what were formerly judgements of identity of content, but also mathematical equalities. Thus, in *Grundlagen*, Frege speaks of "the identity $1 + 1 = 2$", and says that "identities are, of all forms of proposition, the most typical of arithmetic." (p. 69) In the Introduction to *Grundgesetze*, Frege explicitly acknowledges this move:[33]

> Instead of three parallel lines I have adopted the ordinary sign of equality, since I have persuaded myself that it has in arithmetic precisely the meaning that I wish to symbolize. That is, I use the word "equal" to mean the same as "coinciding with" or "identical with"; and the sign of equality is actually used in arithmetic in this way.

As Frege now sees matters, with a rather revisionist ring given his own prior view, the notion that mathematicians (including himself) actually had in mind when they employed the equality symbol was identity, and it is this "ordinary sign" that subsumes identity of content. But although like equality it stands for an objectual relation, it is not a symbol peculiar to mathematics, as is equality; rather it is a general symbol, applicable to propositions about all sorts of things, as is identity of content.[34]

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[33] *Grundgesetze*, p. 6.

[34] Fifty years later, Tarski, in his *Introduction to Logic* (New York: Oxford University Press, 1951), still felt the need to remark on these matters, and devoted an entire section (§19) to discussion of "mathematicians who - as opposed to the standpoint adopted here - do not identify the symbol "=" occurring in arithmetic with the symbol of logical identity." (p. 61) For them, equality is a "specifically arithmetical concept." The problem Tarski sees with this view is that there is a breakdown in the generality

As we have portrayed matters, the primary motivation for Frege to seek unification arises from general considerations of logic that arise narrowly with respect to the formal system of the *Begriffsschrift*. With *Grundlagen*, however, the picture considerably widens, with the emergence of Frege's logicism, the mathematical agenda that was to inform all the remainder of Frege's work. In *Grundlagen*, Frege took on the task of showing something substantive about logic, that abetted with appropriate logical definitions, arithmetic could be reduced to it. Moreover, this reduction would be sufficient to establish that arithmetic truths are analytic truths, since they could be proven solely from logical laws and the definitions. Carrying through this program, however, required clarifications of the logic; things had to be made clear which prior considerations had not shined sufficient light upon. In particular, given the manner in which Frege undertook to define (cardinal) number, identity statements became essential. The reason for this Frege lays out in one of the most famous passages of the *Grundlagen*, opening §62:

> How, then, are numbers given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a proposition that words have any meaning, our problem becomes this: To define the sense of a proposition in which a number word occurs. That, obviously, leaves us still a very wide choice. But we have already settled that number words are to be understood as standing for self-subsistent objects. And that is enough to give us a class of propositions which must have a sense, namely those which express our recognition of a number as the same again. If we are to use the symbol *a* to signify an object, we must have a criterion for deciding in all cases whether *b* is the same as *a*, even if it is not always in our power to apply this criterion.

Frege's reasoning in this remark begins with what he takes to have been shown to this point in *Grundlagen*, that numbers are logical objects, and

of substitution; for mathematics "it becomes necessary to give a special proof that this replacement is permissible in each particular case it is applied." (p. 61) he makes a query with respect to it. His answer is to invoke the context principle - access to such objects is only through the truth of propositions about them - and to single out a certain sort of proposition - identity statements - as central, for in order to know what sort of object a number is, we must know the identity conditions that obtain for them; that is, the conditions under which identity statements about numbers are true. These Frege gives by *Hume's Principle*:

The number of *F*s = the number of *G*s iff *F* and *G* are equinumerous.

It is necessary in order for something to be a number that it satisfy the criterion of identity given by Hume's Principle.

But is it also sufficient? In the ensuing sections, through §69, that form much of the heart of the *Grundlagen*, Frege seeks to answer this question by exploring whether Hume's Principle can stand as a contextual definition of the numerical operator.[35] The answer he ultimately gives is that it cannot; the reason is what is known as the "Julius Caesar" problem. The problem is as follows: If a criterion of identity is to serve as a contextual definition, it must obtain in *any* identity statement in which the numerical operator occurs. However, as given by Hume's Principle, the criterion only applies if a number is given in just this way; consequently, it is undefined in "The number of *F*s = Julius Caesar." But lacking a way of knowing whether this is true or false, we are left short of the generality required of definition. Frege's response is to limit the cases by giving an explicit definition of the numerical operator that entails only the relevant identity statements; i.e. that entails Hume's Principle. So although Frege backs away from construing Hume's Principle as a contextual definition, he still holds that it states a condition that must be met as part of the characterization of number.

The interest to us of Hume's Principle is that *qua* identity criterion it states conditions on the truth of *identity statements*. According to the context principle, in order to gain access to logical objects such as numbers, we must be able to form propositions that are

[35]More precisely, he examines what he takes to be a comparable condition for the direction operator.

about such objects. If these are to be propositions about their identity, then the identity relation must hold of the objects; it *must* be objectual identity. What will not suffice are propositions about two expressions, that they refer to the same object. Even though "$a \equiv b$" is true if and only if "$a = b$" is, the former could only be used to give a *criterion of identity* for numerals, not for numbers; the point is to give criteria of identity for objects, not a criteria for coreference of the ways of designating objects.[36] Objectual identity was thus the only notion of identity that would do for purposes Frege now has in mind, the objectual characterization of number.

In the central sections of *Grundlagen*, before arriving at his negative conclusion, Frege defends the proposed contextual definition specifically with respect to that part of it stated as an identity statement. In doing so he explicates the identity notion he has in mind in two crucial ways. First he specifies the content of such statements:[37]

> Our aim is to construct the content of a judgement which can be taken as an identity such that each side of it is a number. We are therefore proposing not to define identity specially for this case, but to use the concept of identity, taken as already known, as a means for arriving at that which is to be regarded as identical.

Since numbers are objects for Frege, identity here is *objectual* identity in the most general sense, the relation that any object whatsoever bears to itself. Second, Frege elucidates that statements of objectual identity play the logical role expected of identities; they do so because Frege takes this logical role to be the defining characteristic of identity. So Frege writes in *Grundlagen*, §65 that he adopts Leibniz's "definition of identity" in terms of substitution *salva veritate* - "Things are the same as each other, of which one can be substituted for the other without loss of truth" - elaborating that "in universal substitutability all the laws of identity are contained." (p. 77). What we thus observe with these re-

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[36]Of course this would be fine if numerals were numbers; that is, if one adopted the formalist perspective on number. Frege, however, was consumed with scorn for this view; cf. discussion below.

[37]*Grundlagen*, p. 74

marks is the emergence of Frege's "mature" view that he would hold consistently throughout his subsequent work: identity statements are statements of objectual identity, such that in their presence in proof substitution is validated.

The way Frege puts matters in the *Grundlagen* is of course quite informal, with vestiges, in the way he talks of substitution, of the logical literalism that so strongly colored his presentation in *Begriffsschrift*. Frege shakes this off, however, when he comes to giving his formal presentation in the first volume of *Grundgesetze*, nine years later; there, for the first time, an identity theory is stated in a recognizably modern fashion. Unlike in *Begriffsschrift*, in which identity of content was introduced by definition, the identity symbol in *Grundgesetze* is an undefined term; what are specified are the truth-conditions of statements in which this symbol occurs.[38] These Frege gives in §7: ""$\Gamma = \Delta$" shall denote the true", he writes, "if Γ is the same as Δ, in all other cases it shall denote the false." Also unlike *Begriffsschrift*, Frege no longer speaks of the replacement of symbols, when in §20 he specifies the logical role of identity statements, via Basic Law III:[39]

$$\vdash (a = b) \supset (f(a) \supset f(b)).$$

Frege describes Basic Law III in the following way:[40]

> If $\Gamma = \Delta$ is the true, then $[\forall f\,(f\,(\Delta) \supset f\,(\Gamma))]$ is also the True; i.e., if Γ is the same as Δ, then Γ falls under every concept under which Δ falls; or, as we may also

[38]"Truth-conditions" are not meant here in the Tarskian sense; Frege does not have truth-conditions in this sense. Rather what is meant are the conditions under which a sentence denotes the True.

[39]What we give is actually what Frege labels Basic Law IIIa in §50, which he derives from the generalized Basic Law III by instantiation of quantifiers. In this section, Frege proves a number of consequences of Basic Law III, including the law of self identity, i.e.:

$$\vdash (a = a).$$

[40]*Grundgesetze*, p. 71.

say: then every statement that holds for Δ holds also for Γ.

And finally, unlike in *Begriffsschrift*, Frege explicitly assumes in *Grundgesetze* a generalized notion of denotation, allowing mathematical equality, along with identity of content, to be subsumed under (objectual) identity, for now a number can be denoted by its numeral and by other complex expressions that can be stated in the conceptual notation. Not only is the numeral "5" the (atomic) name of a number, but so is the complex name "2+3": "2+3 = 5" is a true *identity statement* because both "2+3" and "5" denote the number five.

From a purely logical perspective, all is now in place as far as identity is concerned, unified with respect to objectual identity. But yet there still remains for Frege a substantial *semantic* issue about identity statements that he must address if he is to defend his view of number against its detractors.

7. As noted, in *Grundgesetze*, Frege specifies his understanding of equalities as objectual identity statements; to repeat the remark from the Introduction to *Grundgesetze* cited above, he says: "I use the word "equal" to mean the same as "coinciding with" or "identical with"; and the sign of equality is actually used in arithmetic in this way." In a contemporaneous letter to Peano, Frege reasserts this view: "I take identity," he says, "to be the meaning of the equals sign." (p 126.)[41] However, in the continuation of the remark from *Grundgesetze*, Frege notes an objection to his view that he curtly dismisses:[42]

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[41]Frege's letter to Peano is undated, but the editors of Frege's *Philosophical and Mathematical Correspondence* place it in the period between 1894 and 1896.

[42]Note that in *describing* an inference there is no bar to *reading* its premisses metalinguistically; interpreting the identity sign in the object language ("=") as objectual identity does not preclude saying metalinguistically that the symbols on either side have the same content. Thus, just as it would be appropriate to describe the implicational statement, which could stand as a premise of a modus ponens inference, by saying: "if a sentence '*a*' implies a sentence '*b*'," so too would it be appropriate to describe the identity statement, which could stand as a premise of a substitution inference, by saying: "if '*a*' and '*b*' have the same content." In places, as in the quotation in the text, Frege avails himself of this way of speaking; other examples are found in *Grundgesetze* §105: "We ourselves use the equality sign to express that the

> The opposition that may arise against this will very likely rest on an inadequate distinction between sign and thing signified. Of course in the equation "$2^2 = 2 + 2$" the sign on the left is different from that on the right; but both designate or denote the same number

But what is this objection to taking equality as identity that rests on such a fundamental confusion, and why is Frege so quick to dismiss it?

The objection is as follows: If equals is identity, then "$a = b$" collapses into "$a = a$," and all arithmetical equations would be trivial. In the letter to Peano, Frege puts it this way:

> What stands on the way of a general acceptance of this view is frequently the following objection: it is thought that the whole content of arithmetic would then reduce to the principle of identity, $a = a$, and that there would be nothing more than boring instances of this boring principle. If this were true, mathematics would indeed have a very strange content.

Frege states the objection again in *Grundgesetze* §138, but this time he gives an explicit reference, quoting a passage from Thomae, his colleague at Jena:[43]

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reference of the group of signs on the left-hand side coincides with the reference of the group of signs on the right," and in "Function and Concept" (p. 22): "What is expressed in the equation '$2.2^3 + 2 = 18$' is that the right-hand complex of signs has the same reference as the left-hand one." But while these comments are in language somewhat reminiscent of *Begriffsschrift*, they are not offered up as analysis, at pain of confusing use and mention.

[43] As far as can be discerned from Frege's writing, it is from Thomae that Frege became explicitly aware of the significance of this issue for his theory of number; there is some issue as to when he became aware. §138 is in Volume II of *Grundgesetze*, dated by Frege October 1902, and published in 1903. Although it is well-known that the publication of *Grundgesetze* in multiple volumes was forced upon Frege by his publisher, given the lukewarm receptions his previous publications had received, (see the remarks by Bynum, *ibid*, p. 34 *ff*), we can definitively place the drafting of §138 as no earlier than 1898, the date of publication of the second edition of Thomae's *Elementare Theorie der analytischen Functionen einer complexen Veränderlichen*, from which Frege drew the passage quoted, which appears on the second page of Thomae's

book. Interestingly, this passage does not appear in the first edition of Thomae's book, published considerably earlier, in 1880. Rather, on the page cited, we find the following remark: "Equating two numbers $n = m$ means either something trivial, namely $n = n$ means that the number n is the number n, or n and m are different in some respect, and only after abstraction do these numbers attain equality from this difference." (p. 2) Although the wording has changed, the content of this passage appears to match, for Frege's concerns, that which he quotes. (Michael Dummett is thus not quite correct when in his *Frege*: *Philosophy of Mathematics* (Cambridge: Harvard University Press, 1991) he says that the first edition of Thomae's book contains all the passages Frege cites from the second edition; cf. footnote 1 on page 241.) Given that Frege cites the first edition of Thomae's book in §28 of *Grundlagen*, published in 1884, we can assume that he was familiar with the issue early on. It is thus plausible that §138 was drafted earlier, contemporaneous with the writing of Volume I., but updated with the newer passage as Frege prepared Volume II for publication. (The only other reference in the section is to a publication of Dedekind's from 1892.) Thomae, incidently, cites the *Grundlagen* on the first page of the 1898 edition of his book, (along with works of Dedekind), only to remark that discussion of them would take him too far afield. Given that both the first volume of *Grundgesetze* and "On Sense and Reference" among other relevant works appeared in the period intervening between the two editions of Thomae's book, and that as colleagues at Jena they apparently had fairly extensive personal discussions, Frege must no doubt have been greatly annoyed at his colleague's simply repeating his argument, without taking account of Frege's response. By 1906, Frege can no longer contain his ire at being ignored, publishing an acidic "Reply to Mr. Thomae's Holiday *Causerie*," in which, among other things, he directly takes on Thomae's views on equality and abstraction (pp. 344-5), and remarks that he is "convinced that with my critique of Thomae's formal arithmetic, I have destroyed it once and for all." This is followed up with an equally acerbic essay in 1908 "Renewed Proof of the Impossibility of Mr. Thomae's Formal Arithmetic." In these essays, Frege's hostility towards the views of his academically more successful colleague is vividly palpable, a reflection of how totally their relationship had soured. While Thomae in 1896 had strongly supported an unsuccessful attempt to advance Frege's career, in 1906 he writes the following about Frege to the university administration: "After all, we only have Colleague Frege left. To my regret I cannot keep secret that his effectiveness has diminished in recent times. The reasons for that cannot be established with certainty. May be one should look for them in Frege's hypercritical tendencies," leading the administration to report that "The Honorarprofessor Hofrat Dr. Frege has probably never been a good docent." At this point, any hope Frege may have had for academic advancement were dashed. We, of course, are quite certain of the problem that affected Frege at this time - Russell's paradox. Thomae apparently did not recognize the cause; it is likely that he had not read *Grundgesetze*, and hence did not comprehend the impact of the paradox, much, one would conjecture, to Frege's annoyance. For a very interesting discussion of the relations of Frege and Thomae, from which the quotations have been drawn, see Uwe Dathe, "Gottlob Frege und Johannes Thomae: Zum Verhältnis zweier Jenaer Mathematiker," in Gottfried Gabriel and Wolfgang Kienzler, eds., *Frege in Jena*:

> . . . Thomae remarks: 'Now if equality or the equality sign = were only to stand for identity, then we would be left with trivial knowledge, or if one prefers, the conceptual necessity *a* is *a* (*a* = *a*)'.

He gives this passage in contrast to Dedekind's view, which embodies three points that Frege "exactly agrees" with:

> (1) the sharp distinction between sign and its reference
> (2) the definition of the equality sign as the identity sign,
> (3) the conceptions of numbers as the reference of number signs, not as the signs themselves

Acceptance of these tenets place Dedekind's - and Frege's - "view in the starkest contrast to every formalist theory, which regards signs or figures as the real objects of arithmetic."[44] The enemy has now been located and the issue joined, and it could not be more fundamental - the nature of number.[45]

Beiträge zur Spurensicherung (Würzburg: Königshausen & Neuman: 1997). (I would like to thank my colleague Christian Werner for helpful discussion and translations.)

[44]Having complimented Dedekind, Frege finishes the section by lambasting him for praising the work of the mathematical formalist Heine.

[45]The formalists were not the only enemies Frege saw to his view of number; casual inspection of the initial sections of the *Grundlagen* reveal empiricism and psychologism as also among their number. My thesis is just that as far as identity statements are concerned, it is the formalists who take center stage. If Frege's goal was to establish that numbers are logical objects, the impediment placed by the formalists was to their objectivity. The only place where Frege directs his remarks on equality to anyone other than the formalists is in his review of Husserl's *Philosophie der Arithmetik* I of 1894, where he remarks that "psychological logicians . . . lack any understanding of identity. This relation cannot but remain perfectly mysterious to them; for if words designated ideas throughout, one could never say '*a* is the same as *b*'; for to be able to say this, one would first have to distinguish *a* from *b*, and they would then just be different ideas." (p. 200.) Frege continues this passage by stating his agreement with Husserl that Leibniz's principle of substitution "does not deserve to be called a definition, even if my reasons are different than his. Since any definition

For mathematical formalists (such as Thomae), the conclusion to be reached from the argument cited is that equals is *not* identity. But what then is an equality? The answer they give is that it is a relation between expressions; given their conflation of distinct numerical expressions with distinct numbers, equality is to be understood as an equivalence relation between distinct things. What "2 + 3 = 5" says is that two numbers are numerically equivalent, their equivalence established by abstracting away from where they differ. Given that it also involves an equivalence between unlike terms, one might be tempted here to analogize to Frege's notion of identity of content in *Begriffsschrift*. Doing so however would be to neglect not only that Frege did not take arithmetical equalities as judgements of identity of content, but also a view that persevered throughout Frege's writings, that arithmetical statements, no differently than any other statements, are meaningful statements. In arithmetic, this meaningfulness arises from the reference relation of numerals to numbers, this being an instance of the more general relation holding between signs and objects. Arithmetic, of course, also has its formal side, in the proofs of theorems (about the numbers) from the axioms. But for proof, we need only attend to signs in the formal mode, *qua* symbols, to warrant taking each step from meaningful statement to meaningful statement; what the symbol stands for is not germane for this. Frege is adamant, however, that although proofs are to be understood as symbol manipulations, mathematical concepts are not given by such manipulations. But this is what obtains on the formalists' view of number, as Thomae remarks:[46]

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is an identification, identity itself cannot be defined," and concludes with the remark that "The author's explanation, 'We simply say to any two contents that they are identical if there is an identity between . . . the characteristic marks which happen to be our centre of our interest' . . . is not to my taste." It is the scent of formalism in this remark that Frege surely found distasteful.

[46]Frege quotes this passage, from the second edition of Thomae's *Elementare Theorie der anakytischen Functionen einer complexen Veränderlichen,* in *Grundgesetze,* §88, in critique of Thomae's likening numbers to chess pieces, defined by the role they play in a mathematical game; cf. Thomae's remarks in the section of his *Elementare Theorie* that Frege cites. For discussion of Frege's critique of formalist mathematics found the second volume of *Grundgesetze*, §§88 - 137, see Michael Dummett, *Frege: Philosophy of Mathematics* (Cambridge: Harvard University Pres, 1991), ch. 20, and Michael Resnik, *Frege and The Philosophy of Mathematics* (Ithaca: Cornell University Press,

> It does not ask what numbers are and what they do, but rather what is demanded of them in arithmetic. For the formalist arithmetic is a game with signs that are called empty. This means that they have no other content (in the calculating game) than that they are assigned by their behavior with respect to certain rules of combination (rules of the game).

Thus, that Frege found the formalists' de-trivialization strategy completely unacceptable is none too surprising, as it is ultimately based, in his view, on an incoherent notion of number. Numbers cannot be *just* formal marks, to be manipulated by rule; this would result, as Frege remarks to Peano, in a "chaos of numbers":[47]

> There would not be a single number which was the first prime number after 5, but infinitely many: 7, 8 - 1, (8 + 6): 2, etc. We would not speak of 'the sum of 7 and 5' with the definite article, but of 'a sum' or 'all sums', 'some sums', etc.; and hence we would not say 'the sum of 7 and 5 is divisible by 3'.

Abstraction is no patch, for the notion of numerical equivalence onto which it is to converge must presuppose a notion of number such that distinct numbers, given by distinct numerical symbols, can be the same number. This is to be the accomplishment of abstraction, by which, Frege says, "things are supposed to become identical by being equated." But how can abstraction, a "capability of the human mind" according to Thomae, turn two things into one? - "if the human mind can equate any objects whatever, [abstraction] is especially meaningless, and the meaning of equating will also remain obscure." Frege asks: "What do [the formalists] want to achieve by abstracting? They want - well, what they

1980), ch. 2.

[47]Letter to Peano, p. 127.

really want is identity."[48] To obtain this, however, we do not want formalist obscurity rooted in their confusion of numeral and number; rather, we need, according to Frege, to carefully distinguish symbols from what they stand for; we need to attend to their *semantic* relation. If there can be distinct symbols that are signs for the same thing, it can be said that the "signs '2 + 3', '3 + 2', '1 + 4', '5' do designate the same number."[49] Equality can be construed as identity; the chaos evanesces.

Of course, rejecting the formalist conception of number and accepting Frege's points (1), (2) and (3), does not make the problem raised in Thomae's remark go away in and of itself. What does make it go away is the recognition, all too clear to Frege, that the premise of the argument is simply mistaken - identity statements of the form $\ulcorner a = b \urcorner$ do have "greater cognitive content than an instance of the principle of identity." But whatever this greater cognitive content is to consist in, such that it separates "*a* = *b*" from "*a* = *a*," it cannot be such that it would "prevent us from taking the equals sign . . . as a sign of identity." (Letter to Peano, p. 126.) But what is it that satisfies these dual criteria? It is precisely at this point of the discussion that the notion of *sense* makes its appearance.

8. Let's consider the issue facing Frege. If we harken back to *Begriffsschrift*, where there are two notions, equality and identity of content, Frege only perceived the need to associate modes of presentation with judgements of the latter sort. Frege did this in order to introduce information that was not otherwise specified by the judgement itself as part of conceptual content; modes of determination are called for with

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[48]This and the two previous quotations are from "Reply to Mr. Thomae's Holiday *Causerie*," pp. 344-5, in which Frege, apparently incensed by Thomae, remarks that "After reading what [Thomae] said about abstraction, I vented my feelings in this verse:

> Abstraction's might a boon is found
> While man does keep it tamed and bound;
> Awful its heav'nly powers become
> When that its stops and stays are gone

[49]This passage is from Frege's unpublished "Logic in Mathematics" (p. 224), in which the argument first given in the letter to Peano appears again, although, dated (1914), it was written many years later.

atomic names because all that is indicated as part of conceptual content by their representation in the conceptual notation is their denotation. Things are different in the case of mathematical equality, however; they make no call for modes of determination. This is because all the mathematically significant information is already manifest intranotationally; what else could be relevant to the thought expressed by “2+3 = 5” than just that 5 is the sum 2 and 3? This information, however, is exactly what is obscured if “2+3 = 5” is to be read as an identity statement, for all that would be relevant to its truth would be coreference to the number 5. All we would have would be *denotational* information; what is masked is the mathematical information that the number denoted is the sum of 2 and 3. The tension we see here is nicely put in the following remark by Geach:[50]

> . . . what is conveyed by mathematical equations is the strict identity of what is mentioned on either side of the equation; thus 6:3 = 1+1 because 6:3 is *the* number (not *a* number) which when multiplied by 3 yields the result 6, and 1+1 is that very number. . . What makes the equation informative is that though the same number is mentioned on both sides, it is presented as the value of two different functions - the quotient function and the sum function.

The problem facing Frege is how to capture both of these sorts of information. On the *Begriffsschrift* notion of content, Frege is denied the wherewithal to analyze arithmetic equalities as identities; a general notion of denotation can provide such wherewithal, but then we lose our grip on the mathematical information expressed. But this grip must be maintained, if the criticism of the formalists is to be met.

How then is Frege to capture both sorts of mathematical information? He achieves this by introducing the more fine-grained notion of content embodied in his theory of thoughts, as this is chiefly developed in the early in 1890’s, primarily in the seminal essays, “Function and Concept” and “On Sense and Reference,” and in *Grundgesetze*. This theory is elegant in its simplicity: thoughts are

[50] Geach, *ibid*, p.144.

complexes, the sum of constituent *senses* of which they are composed. Senses are to be characterized functionally; they have the property of *determining* (or presenting) reference (*Bedeutung*). Grouped together to form a thought, they are capable of determining a truth value as reference; but in order to do so, they each themselves must be able to determine a reference. (Their references may be either objects or *n*-level concepts.) Thoughts are *expressed* by sentences (forms of the conceptual notation) in virtue of sentences being composed of *signs*, symbols that express senses. Content, Frege now says in way of contrasting his new view to that of *Begriffsschrift*, "has now split for me into what I call 'thought' and 'truth-value', as a consequence of distinguishing between sense and denotation of a sign." (*Grundgesetze*, pp. 6 - 7)

On this scheme the components of content are related in two distinct way. One is by the semantic relation of determination or presentation of reference by sense; the other is by *judgement*, which is no longer to be understood as a property of an undifferentiated monolithic content. Frege describes judgements in "On Sense and Reference" as "advances from thoughts to truth value," (p. 65), in *Grundgesetze* §5 he says "by a *judgment* I understand the acknowledgment of the truth of a *thought*" and in a letter to Husserl he remarks that "Judgement in the narrower sense could be characterized as a transition from a thought to a truth value." (p. 63). This transition, Frege makes clear is a *cognitive* relation: "*When we inwardly recognize that a thought is true, we are making a judgement*," Frege writes, with the emphasis; he amplifies by remarking that "Both grasping a thought and making a judgement are acts of a knowing subject, and are to be assigned to psychology. But both acts involve something that does not belong to psychology, namely the thought."[51] It is with respect to the status of a thought with respect to its role in judgement that Frege introduces the notion of *cognitive value* - those aspects of judgement that are determinable from the thought itself are the cognitive value of the thought. While Frege does not elaborate very much on the notion of cognitive value, he does make clear one pivotal aspect of the notion. This is that the cognitive value of a thought reflects whether to judge it we must attend to the semantic

[51]The former quotation is from an unpublished manuscript entitled "Logic" of 1897, p. 139; the latter from "Notes to Ludwig Darmstaedter," dated July 1919, p. 253.

connection of *Sinn* and *Bedeutung*. Any thought in which we must take into account the particulars of the determination of reference by the constituent senses in order to advance to truth will have greater cognitive value than thoughts for which this is not required, and any thoughts that so differ in cognitive value will consequently enter into distinct judgements.[52]

With these notions in hand, Frege is now in a position to make his response to the formalist critique, giving an answer to what has come to be known as Frege's puzzle. In *Grundgesetze* §138, immediately following the passage cited in the previous section, in which he quotes Thomae's remark, Frege says:

> The knowledge that the Evening Star is the same as the Morning Star is of far greater value than a mere application of the proposition '*a* = *a*' - it is no mere result of a conceptual necessity. The explanation lies in the fact that the sense of signs or words (Evening Star, Morning Star) with the same reference can be different, and that it is precisely the sense of the proposition - beside its reference, its truth-value - that determines its cognitive value.

The formalists (Thomae in particular) have, according to Frege, just made a "mistake." It is the mistake that arises from not attending to the thought expressed, from an "inadequate distinction between sign and thing signified," the mistake noted in the Introduction to *Grundgesetze*. Interpreting equality as identity *does not* make "*a* = *a*" and "*a* = *b*" have the same cognitive value. It is trivial to judge the truth of the former

[52] Gareth Evans, in discussing Frege in *The Varieties of Reference* (Oxford: Clarendon Press, 1982), gives the following condition on cognitive value: "a sentence *S* has a different cognitive value from a sentence *S*' just in case it is possible to understand *S* and *S*' while taking different attitudes towards them." (p. 19) This condition is undoubtedly false; Evans has confused here expressing different thoughts with having different cognitive value. "Cicero is a Roman" and "Tully is a Roman" on Fregean lights express different thoughts, and one can surely have different attitudes towards them; they do not thereby differ in cognitive value. If two sentences express thoughts that have different cognitive values, then they express different thoughts; the inverse, however, does not hold. Two sentences may express different thoughts, but have the same cognitive value.

given the thought it expresses, for mere inspection of its form shows it to be composed in accordance with a logical law; it is an instance of the law. Its truth is known *a priori*, regardless of the particulars of how the constituent senses determine reference.[53] It is just such particulars, however, that we need to regard in order to judge the truth of "*a* = *b*"; because "*a*" and "*b*" *can* have different senses, in order to make a judgement we must ascertain whether they determine the same reference. And because of this, "*a* = *b*" has greater cognitive value than "*a* = *a*" and must express a different thought; since the latter is a "mere result of a conceptual necessity," the latter, *a fortiori*, cannot be. With this insight, Frege has answered the challenge of the formalists.

The argument Frege gives here would appear to be completely general; whether a thought is trivial has to do with its composition, and not with what sorts of things "*a*" and "*b*" stand for. But yet again, as in *Begriffsschrift*, Frege turns in the midst of a discussion of the "basic laws of arithmetic" to a non-arithmetical illustration of this point. Such examples are employed in support of his position not only in *Grundgesetze*, but also in his other remarks on identity statements of this period. In the undated letter to Peano, in his remarks "On Mr. Peano's Conceptual Notation and My Own" (1897), in an unsent letter to Jourdain (1914), in "Function and Concept" (1891), and in "Logic and Mathematics" (1914), perceptual examples are employed, whether along the lines of the famous case of the "The Evening Star is the Morning Star," or of Astronomer X's and astronomer Y's comets that turn out to be the same (in "On Mr. Peano's Conceptual Notation and My Own"), or the case of explorers who see a mountain from different directions, the "Afla" and "Ateb" of the letter to Jourdain. But such examples are not used to the exclusion of specifically arithmetical examples; these are found in the letter to Peano, "On Mr. Peano's Conceptual Notation and My Own," "Function and Concept," in a letter to Russell (1904), and

[53]For Frege, while "*a* = *a*" is trivial, it remains sensible; it expresses a thought. This was not held by everyone. Moore, for instance, remarks that "A sentence of the form " — is identical with —" *never* expresses a prop[osition] unless the word or phrase preceding "is identical with" is different from that which follows it." (G. E. Moore. "Identity" in *Commonplace Book: 1919 - 1953*, edited by Casimir Lewy, (London: George Allen and Unwin, 1962.)

perhaps most clearly in “Logic and Mathematics”:[54]

> . . . one cannot fail to recognize that the thought expressed by ‘5 = 2 + 3’ is different than that expressed by the sentence ‘5 = 5’, although the difference only consists in the fact that in the second sentence ‘5’, which designates the same number as ‘2 + 3’, takes the place of ‘2 + 3'. So the two signs are not equivalent from the point of view of the thought expressed, although they designate the very same number. Hence I say that the signs ‘5’ and ‘2 + 3’ do indeed designate the same thing, but do not express the same *sense*. In the same way ‘Copernicus’ and ‘the author of heliocentric view of the planetary system’ designate the same man, but have different senses; for the sentence ‘Copernicus is Copernicus’ and ‘Copernicus is the author of the heliocentric view of the planetary system’ do not express the same thought.

In this passage, Frege’s last known remark on identity statements, he clearly indicates the intended generality of the method; it is to apply in a uniform fashion to all cases, regardless of whether an identity statement is arithmetic or not. Not only is “2 + 3 = 5" to be taken as a statement of identity, so too is “Hesperus is Phosphorus” or “Copernicus is the author of the heliocentric view of the planetary system”; in the latter examples, “‘is’ is used like the ‘equals’ sign in arithmetic, to express an equation.” (“On Concept and Object,” p. 44.)[55] By looking at the thought expressed, at the way that it is internally composed, the cognitive value of an identity statement can be assessed independently of the subject matter of the statement. Thus, if we were to take arithmetic statements as analytic, as Frege does in *Grundlagen*, the difference in the relation of thought to judgement bifurcates among them in the same way as it does

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[54]“Logic and Mathematics,” p. 225

[55]Frege, however, does not pursue the analysis of identity statements in “On Concept and Object,” (the sense/reference distinction is not introduced there). Rather, his remarks are in the context of an admonition to distinguish this case from “is” when used in predications that denote concepts, as in “Phosphorus is a planet.”

for non-arithmetical statements, even though for the latter the distinction in cognitive value would cleave the same distinction as the Kantian distinction of analytic and synthetic.[56]

Let us take stock at this point. The issue of the mathematical information expressed by "2+3 = 5" now comes down to distinguishing its sense and reference, and the relation thereof. In particular, the thought this sentence expresses contains as a constituent sense the sense of "2+3" - that the sum-operation is applied to 2 and 3 - that determines 5 as the reference of "2+3," the same number that is the reference of "5." It is thus the thought expressed by "2+3 = 5" that carries the mathematically significant information, not the reference. (Assuming the latter is the mistake at the heart of the formalists' critique.) The *way* it carries it, via two distinct senses, determines its cognitive value as greater than that of the thought expressed by "5 = 5." To contrast with *Begriffsschrift*, what Frege has done has taken the content of judgements of equality and transformed it into the thought expressed by a mathematical identity, including the structure discernable in the form of the judgement, "distinguish[ing] parts in the thought corresponding to the parts of a sentence, so that the structure of the sentence serves as an

[56]In "On Sense and Reference," Frege remarks that "$a = a$" is "according to Kant, is to be labelled analytic,", but he refrains from labeling "$a = b$" synthetic, remarking only that it "cannot always he established *a priori*." (p. 56) Interestingly, it is Kant's sense of analytic he refers to here, by which Frege means logically trivial, and *not* his own notion, which he had gone to great lengths to establish in *Grundlagen*. In fact, the analytic/synthetic distinction plays little role, if any, in Frege's thinking during the period we are discussing; hence the tempered nature of the remark in the last sentence of the text. Thus, in *Grundgesetze* Frege speaks of the goal of his project as showing that arithmetical truths are logical, not analytic, truths. While there is much to be said about this, briefly the reason for this is that given the primacy of thought, (as opposed to its derived status in *Begriffsschrift* and *Grundlagen*), at this point Frege can say little more than that arithmetic truths are analytic because they are about logical objects; while synthetic truths are about non-logical objects. This is all that could be discerned from the information contained in the thought itself. But this would not be a very robust notion of analytic and synthetic, and certainly would not satisfy anyone looking for an explication of the Kantian notions. But of course the theory of thought otherwise pays large dividends for Frege, sufficiently so that he could abandon an objective that had previously so animated his thinking. (I am indebted to Aldo Antonelli for discussion of this point.)

image of the structure of the thought."[57] If this imaging is isomorphic, as in "2+3 = 5," the structure of the thought expressed will be no more complex than that of the sentence itself. Isomorphism fails in one central case, however; with atomic names the structure of the thought will be more complex than that of the sentence.[58] The source of this complexity is where Frege finds common ground with his earlier views; it is with the modes of determination (*Bestimmungsweise*) of *Begriffsschrift*, now wrapped up in the more general notion of sense.[59] Thus by parity, along with the generalization of denotation from atomic to complex terms goes the generalization of modes of determination; what had applied to a narrower class of terms now applies inclusively to the larger class. But this common germ should not occlude the changed role of this notion in

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[57]"Compound Thoughts," p. 55. Here Frege clearly reveals a *syntactic* view of thoughts; he continues the remark above by saying "To be sure, we really talk figuratively when we transfer the relation of whole and part to thoughts." This raises issues about the structure of thoughts; for example, are there parts of thoughts that don't express senses, that serve as "connecting tissue," holding thoughts together as wholes, (as we might think of syncategoremic expressions doing in syntax)? If there are, can they have an effect on the thought expressed. This relates to the issue of whether we can have equivalent ways of composing a given sense; cf. discussion of the status of Basic Law V in §9. (Again, thanks to Aldo Antonelli for discussion of this point.)

[58]Note that in order to make the argument for greater complexity of thought we need a language in which there are *multiple* atomic names with the same denotation; thus again arithmetic falls short.

[59]There are differences between the notions however that we must observe. In *Begriffsschrift*, what was determined was conceptual content; modes of determination are of an object and are associated with terms. Senses, on the other hand, are themselves objects, albeit abstract and logical, that are expressed by terms; now the determination itself, as a constituent part of sense, is *part* of the content. The correspondence of *Bestimmungsweise* and *Sinn* was first noted by Ignacio Angelelli, in *Studies on Gottlob Frege and Traditional Philosophy*, (Dordrecht: D. Reidel, 1967), pp. 38 - 40, and subsequently has been remarked upon by Terrell Ward Bynum "Editor's Introduction," in Gottlob Frege, *Conceptual Notation and Related Articles*, translated and edited by Terrell Ward Bynum, (Oxford: Clarendon Press, 1972), pp. 65 - 68; Richard Mendelsohn, "Frege's *Begriffsschrift* Theory of Identity," *Journal of the History of Philosophy*, 22, 3, (1982), 279 - 99: Peter Simons, "The Next Best Thing to Sense in *Begriffsschrift*", in J. Biro and P. Kotatko, eds., *Frege: Sense and Reference One Hundred Years Later*, (Dordrecht: Kluwer, 1995); and Michael Thau and Ben Caplan, "What's Puzzling Gottlob Frege," ms, UCLA, 1999.

the direction of explanation. In *Begriffsschrift*, Frege begins with a notion of judgements as forms and appeals to modes of determination in order to explain their connection to thought. In his mature work, explanation goes the other way round. Frege begins with thoughts, and explains by appeal to their constituent senses their connection to judgement. What Frege believes he obtained from looking at matters this way was a defense of his view of number. Numbers are logical objects, and true identity statements about them are logical truths; but this does not leave them bereft of mathematical content, given the theory of thought. And it was this that was the pay-off of his new perspective on identity statements.[60]

9. Central of the thesis we have been exploring is that in order to fully comprehend why Frege introduces the *Sinn/Bedeutung* distinction we must comprehend the role it plays in *defense* of his theory of number. But it is important to bear in mind that the distinction itself is not significant for the *construction* of that theory; the sense/reference distinction plays no role in that. The construction does, however, exploit the theory of identity as part of specifying what sort of logical objects numbers are; for Frege in *Grundgesetze* numbers are extensions of certain concepts, and the theory of identity is needed to characterize the criteria of identity for such extensions. As was discussed in §6, Frege's approach is anticipated in *Grundlagen*, but it is in the *Grundgesetze* (and in an introductory manner in "Function and Concept") that Frege thought he had suffi-

[60] It is of interest to compare at this point remarks of Ramsey's in "The Foundations of Mathematics," (in Frank P. Ramsey, *The Foundations of Mathematics and other Logical Essays*, (New York: The Humanities Press, 1950), pp. 16-17.) Observing that "in '*a* = *b*' either '*a*', '*b*' are names of the same thing, in which case the proposition says nothing, or of different things, in which case it is absurd. In neither case is it the assertion of a fact. . . . When '*a*', '*b*' are both names, the only significance which can be placed on '*a* = *b*' is that it indicates that we use '*a*', '*b*' as names of the same thing or, more generally, as equivalent symbols." Thus, Ramsey's response is to entertain a view reminiscent of that of *Begriffsschrift*, but unlike that view, it is strictly a relation between symbols, and does not encompass the subtlety of Frege's modes of determination. Ramsey briefly explores whether such a construal of identity statements could suffice for mathematics; concluding in the negative, he rejects the account, (as well as Russell and Whitehead's account in *Principia Mathematica*), and endorses the view of Wittgenstein in the *Tractatus* that dispenses with an identity symbol in favor of the convention that each distinct symbol has a distinct meaning (reference). Cf. pp. 16-20, 29-32

ciently clarified his views so that he could properly address the issue. It was central to Frege's view of identity in this period that in conformance with his insistence on a clear distinction between function and object, it is only objects that fall under the identity relation, not functions. Identity is to be construed strictly as objectual identity. However, what is to be taken as an object is considerably elaborated in *Grundgesetze*, to include two sorts not introduced previously, truth-values and courses-of-values.[61] Taken in coordination with the articulation of levels of functions, this reification provided, Frege thought, the tools for putting into place the foundations of the notion of number, through the statement of the infamous Basic Law V. The sad story of Basic Law V is well-known, and I will not rehearse here how it paved the road to paradox. It is, however, worth pausing to consider its formulation, for it will allow us to get a complete picture of Frege's mature theory of identity.

In *Grundgesetze*, Frege propounds the theses that (i) truth-values are a type of logical object, and (ii) sentences are complex names, having truth-values as their references. Given the generalized notion of denotation in *Grundgesetze*, the syntax allows for identity statements such as (I):

Russell was English = Frege was German (I)

For an identity statement like (I) to be true, the complex terms that stand to the sides of the identity sign must refer to the same thing, either both to the True or both to the False, (although of course their senses differ). Thus, under the assumptions (i) and (ii), Frege was able to utilize the identity sign as a convenient way of expressing material equivalence.[62]

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[61] In *Grundgesetze*, §10, Frege shows that truth-values can be reduced to certain courses-of-values; so taken, Frege only introduces one novelty to his ontology, not two. See footnote 64.

[62] In "Boole's logical Calculus" (1880/81) Frege makes clear that the material biconditional is to be expressed by the conjunction of conditionals; unlike his identity sign, "Boole's identity sign," Frege writes, "does the work of my conditional strokes: $B \supset A$ and $A \supset B$." (p. 37) In *Begriffsschrift* §3, Frege says what it would be for two judgements to have the same conceptual content - "the consequences which can be derived from the first judgement combined with certain others can always be derived from the second judgement combined with the same others". Frege gives this condition in the service of elucidating what are irrelevant differences in representation with

From the standpoint just elaborated, notice that what we would say about (II) is just the same as (I); it too is true just in case what stands on either side of the identity sign refer to the same truth value:

$$\text{Max is a renate} = \text{Max is a cordate} \qquad \text{(II)}$$

There is something more, however, that we would like to say about (II) than what we also say about (I); (II), unlike (I), contains equivalent predicates, for every renate is a cordate, and every cordate a renate, something we can state by (III):

$$\text{Renates} = \text{cordates.} \qquad \text{(III)}$$

(III), it would appear, equates concepts; but if so, then according to Frege, it is not well-formed, for an identity statement can only contain names of complete (saturated) objects besides the identity symbol:[63]

> the relation of equality, by which I understand complete coincidence, identity, can only be thought of as holding for objects, not concepts. . . . we may not write $\Phi = X$, because here the letters Φ and X do not occur as function-letters. But nor may we write $\Phi(\ ) = X(\ )$, because the argument-places have to be filled.

"An isolated function-letter without a place for an argument," Frege says, "is a monstrosity." (*Grundgesetze*, §147.) Thus, insofar as there is a relevant parse of (III), it is as "( ) is a renate = ( ) is a cordate;" but this could only be thought to be well-formed if one were to confuse concepts and objects, what is unsaturated with what is saturated. We are

respect to the conceptual notation. But what Frege does not provide in *Begriffsschrift* is the syntactic wherewithal to *state* such identity; judgements of identity of content only hold with respect to terms that denote their conceptual contents. Thus, we don't have any counterpart to (I); i.e. we don't have "Russell was English ≡ Frege was German." But even if we did, it could not the work of (I), since it is false. "Russell was English" plainly does not have the same conceptual content as "Frege was German," for one contains Russell, the other Frege.

[63] "Comments on Sense and Meaning," p. 120 - 1.

not, however, at a complete loss here, for the equivalence of concepts can be specified under a weaker condition, that the same objects fall under each concept. Concepts that fall under the complex second-level relation:

$$\forall a\ \Phi(a) = \Psi(a),$$

will meet this condition. The resulting generalization:

$$\forall a\ f(a) = g(a),$$

is not itself an identity statement - the identity sign as it occurs here is part of a complex second-level relation, under which first-level concepts fall - although its truth depends upon identity statements like (II), whose terms denote truth-values, being true.

The importance of being able to specify such a relation for concepts is that it paved the way for Frege to introduce an important novelty of *Grundgesetze*, the introduction of terms referring to *courses-of-values*, of which extensions are the special case for concepts. Frege was insistent on the view that extensions are logical objects; it is a claim that stands at the heart of his logicism, for in *Grundgesetze*, Frege explicitly defines numbers as the extensions of certain concepts. Extensions themselves (and courses-of-values in general), however, are not defined, but rather are legitimately introduced by satisfying a criteria of identity. Frege's idea here was simply this: If we ask what sort of objects extensions are, it is sufficient to answer that they are the sort of objects that are the same just in case the same objects fall under the corresponding concepts.[64] What constitutes a corresponding concept is

[64]Frege recognized that the sufficiency of this approach may not be obvious, especially in light of the Julius Caesar problem that had led him to reject the contextual definition of number in *Grundlagen*, and he devotes §10 of *Grundgesetze* to this matter. Frege thought that with his method of introducing extensions in *Grundgesetze*, which is not by contextual definition, he could overcome the problem. The problem as now conceived is that to ask whether an arbitrary object *p* is a course of values is reduced to whether "$\dot{\varepsilon}f(\varepsilon) = p$" is true. This would only result in the appropriate class of identity statements because Frege thought that he could characterize all objects as courses-of-values. This cannot work, however, for if it did, there would be a consistency proof for the system of *Grundgesetze*. (Thanks to Aldo Antonelli for bringing this to my attention.) For discussion of this section of *Grundgesetze*, one of the most complicated

made explicit by the composition of complex names of extensions. As Frege notates, "$\dot{\varepsilon}f(\varepsilon)$" refers to an extension; then the first-level function $f(\xi)$ that stands as the argument of the second-level function $\dot{\varepsilon}\psi(\varepsilon)$ is the corresponding concept. Thus, two extensions are the same:

$$\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha),$$

if and only if the concepts to which they correspond are identical. Concepts are identical if and only if they fall under the second-order relation above; that is, if the same objects fall under them. Identity of extension is therefore to be equated with the extensional equivalence of the concepts; the result is:

$$(\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\forall \mathfrak{a}\ f(\mathfrak{a}) = g(\mathfrak{a})).$$

Frege remarks: "We can transform the generality of an identity into an identity of courses-of-values and vice versa. This possibility must be regarded as a law of logic, a law that is invariably employed, even if tacitly, whenever discourse is carried on about the extensions of concepts." (*Grundgesetsze*, §9.) This "law" of logic is Frege's Basic Law V. Frege thought at the time of the *Grundgesetze* that because Basic Law V was a logical law, that extensions being logical objects was secured, since they satisfy a logical criteria of identity, and thus that his theory of number was justified. But, of course, Frege was wrong about this, as he instantly understood upon learning of Russell's paradox, and the inconsistency of Basic Law V.

It should be clear that as we have described the role of Basic Law V in Frege's mature theory of number, we have had no recourse to the sense/reference distinction, even though it takes the form of an identity statement. This is not particularly surprising, given that Frege's goal was to establish extensions as references. But we may ask a further question about Basic law V, that arguably does implicate the sense/reference distinction. Frege remarks at various points that in order for a

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and difficult in the book, see Michael Dummett, *Frege: Philosophy of Mathematics*, p. 209*ff*, Michael Resnik, *Frege and The Philosophy of Mathematics*, p. 208*ff* and A. W. Moore and Andrew Rein, "*Grundgesetze*, Section 10" in L. Haaparanta and J. Hintikka, eds., *Frege Synthesized: Essays ont he Philosophical and Foundational Work of Gottlob Frege* (Dordrecht: D. Reidel, 1986).

statement to qualify as a logical law it must at least meet two standards, that it be a logical truth, and that it be self-evident. Frege, however, is somewhat foggy about what constitutes satisfying these conditions, but a natural initial thought, given the notions at hand, would be to look at the intensional structure of the statement of the law, and appeal to synonymy of the terms. Indeed, this is where Frege initially glances; in "Function and Concept" he identifies the thoughts expressed by the two statements joined in Basic Law V. Each "expresses the same sense, but in a different way," he says. (p. 27.) No doubt Frege's thinking here is that if Basic Law V is a logical law then it must at least be distinguished from a garden variety material equivalence like (I). What he wants to capture is that, in essence, the identity of extensions is no different than the identity of the corresponding concepts; *they* are one and the same circumstance, (just as, for instance, that lines are parallel is the same as their having the same direction). The theory of the *Grundgesetze*, however, only has identity of reference and identity of sense; it provides no way of directly expressing this other sort of equivalence relation. It is, however, *entailed* by identity of sense, and this is Frege initial conjecture about Basic Law V.

If Basic Law V is justified as a logical law by its intensional structure, then it would appear that the sense/reference distinction *is* implicated, at a very fundamental level in Frege's theory, for if the statements that stand as the terms of Basic Law V express the same thought, then Basic Law V is analytic. But, if they do express the same thought, it is not in a way that satisfies Frege's other criterion for logical laws, that they be self-evident. Frege allowed that there may be various overt forms that may express a single thought. For example, active sentences and their passive counterparts are formally distinct, yet arguably the transformation between them leaves untouched the senses that compose the thought expressed. There is only one thought expressed, composed of the same senses. Statements so related Frege called *equipollent*; as Frege observes, because equipollent statements express the identical thought, to recognize the thought expressed by one statement is to recognize the thought expressed by the other.[65] The statements that stand as the terms of Basic Law V, however, are not equipollent. They contain expressions that refer to distinct second-level

[65] Cf. "A Brief Survey of My Logical Doctrines," pp. 197 - 8.

functions, and given that distinctness of reference entails distinctness of sense, the thoughts they express therefore must contain different senses. If thoughts are to be defined as the composition of their constituent senses, there is no guarantee that by recognizing a thought composed in one way, that we can also recognize it composed in some other way. But then there is also no guarantee that Basic Law V is self-evident.[66]

The problem here is that there is really no wiggle room in Frege's theory to have thoughts be the same, yet composed of different senses; no doubt it would have struck Frege that this does not square with his other assumptions about sense and reference. What is consistent with these views is that the terms of Basic Law V do not express the same thought, a view buttressed by the observation that:

$$((\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = ((\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)),$$

contrasts in cognitive value with Basic Law V just as "Hesperus is Hesperus" contrasts with "Hesperus is Phosphorus," or "5 = 5" with "2+3 = 5," implying that the terms express different thoughts. Frege's reaction, however, was somewhat different; he retreats to an extensionalist position, which he maintains throughout *Grundgesetze*. He tells us during this period that "reference and not the sense of words [are] the essential thing for logic . . . the laws of logic are first and fore-most laws in the realm of reference and only relate indirectly to sense."[67] In his review of Husserl's *Philosophie der Arithmetik* in 1894, Frege elaborates:[68]

This reveals a split between psychologistic logicians

[66]We should be careful here to distinguish the issue whether two distinct complex terms can express the same thought from abbreviatory definition, in which the sense of non-complex term is equated with that of a complex term. Atomic terms may have compositionally complex senses; this composition is just not transparently revealed as it would be with non-atomic terms.

[67]"Comments on Sense and Meaning," p. 122. This paper is dated by the editors of Frege's *Posthumous Writings* as 1892 - 1895. While the editors adopt the convention of translating *Bedeutung* as "meaning," I aver to the more common "reference" in the quotation.

[68]Review of Husserl, p. 200.

> and mathematicians. What matters to the former is with the sense of the words, as well as the ideas which they fail to distinguish from the sense; whereas what matters to the latter is the thing itself: the reference of the words.

Thus, regardless of whether the two sides of Basic Law V have the same sense or different, they determine the same reference, and that is all that matters for logic. In particular it doesn't matter to the characterization of numbers as logical objects; all Frege needs to characterize the logical role of Basic Law V is the weaker notion of material equivalence. But although taking an agnostic view, Frege apparently retained his qualms about the issue left open. In a well-known remark, Frege, in commenting upon Russell's Paradox, says about Basic Law V that "I have never concealed from myself its lack of the self-evidence which the others possess, and which must properly be demanded of a law of logic." (*Grundesetze*, Appendix II.) But there is no reason to think that Frege thought that this self-evidence would have been found through reflection on the sense/reference distinction; as Frege saw the issue it cut much deeper, to the heart of "how numbers can be conceived as logical objects." We know now that it cut to a point where it is insuperable.[69]

[69]The position we have developed in the last paragraphs echoes that of Michael Dummett; see his *Frege*: *Philosophy of Mathematics* (Cambridge: Harvard University Press, 1991), especially pp. 168 - 176, and *The Interpretation of Frege's Philosophy*, *ibid*, especially pp. 529 - 537. In the latter book, and in *Frege and Other Philosophers, ibid*, Dummett mounts an extensive critique of Hans Sluga, who is the chief proponent of identity of sense view; see his *Gottlob Frege* (London: Routledge and Kegan Paul, 1980), especially pp. 149 - 157, and "Semantic Content and Cognitive Sense" in L. Haaparanta and J. Hintikka, eds., *Frege Synthesized: Essays on the Philosophical and Foundational Work of Gottlob Frege* (Dordrecht: D. Reidel, 1986). Sluga states (in the latter article) that Frege "initially introduced the notion of sense in order to explain why Axiom V is not a synthetic truth." (p. 61.) His argument for this view stems from the claim that Frege's *Begriffsschrift* theory only applied to synthetic truths, so that "Frege is no longer able to explain how any but the most trivial arithmetical equations can turn out to be logical truths." (p. 58.) To do this, Sluga argues, Frege needed the sense/reference distinction in order to countenance analytic truths through identity of sense. In maintaining that the terms of Basic Law V express the same thought, Sluga appears to deny that they have different component senses when he remarks that "It is only our subjective perception and our manner of speaking that distinguish the statement about functions from that about value-ranges . . . a thought concerning a

10. While one can admire the elegance and coherence of Frege's account, there is one thing missing. There is no argument for the central concept: sense. Why should we think that such things exist, with the properties that Frege ascribes to them? The importance to Frege of providing such an argument is clear, for he does not wish to be accused of introducing some unreal, mystical notion just in order to avoid a foundational problem with his view of number. The remedying of this defect is the task that Frege takes on in his seminal essay on the topic of identity statements, "On Sense and Reference."

How is Frege to establish the reality of sense? He does this by arguing, at length and with great force, that senses can be *Bedeutung*, that they can be references. Since senses can be referred to, they must exist; given Frege's realism about abstract objects, senses can be references in just the same way that numbers can be. Looked at this way, the central goal of "On Sense and Reference" is to give an "existence proof" of senses. Frege develops this with great care, choosing his words judiciously; since this essay appeared in a general, non-technical philosophical journal, Frege does not want his audience to view him as pursuing an arcane issue in mathematics. The difficulties Frege faced with publication are well-known.[70] Frege, consistently stung by the limited, yet highly critical reception of his work among his contemporaries, appeared to feel that he had never achieved the proper "voice" for expressing his views, and hence vacillated between more and less technical presentations of his work. This was particularly acute around the time of the publication of "On Sense and Reference," in the early 1890's, for this is when *Grundgesetsze* was deemed too technical to be published in one volume.[71] As with *Grundlagen* with respect to

function is the same as one concerning a value-range." (*Gottlob Frege, ibid,* p. 157.) It is difficult to see how to reconcile this with the fact that different references entail different senses, unless it is maintained that the second-level concept-terms in Basic Law V refer to the very same concept. For this and other reasons, Sluga's rendering strikes one as a most implausible reading of Frege.

[70]This is well-chronicled in Bynum's "On the Life and Work of Gottlob Frege," *ibid.*

[71]In unpublished "Comments on Sense and Reference," Frege elaborates on the issue discussed in the previous section, remarks that are missing, we might conjecture, from "On Sense and Reference" at least in part because of the technical notation required to make the point; indeed, he remarks that he will elaborate his view of concepts "for

Begriffsschrift, "On Sense and Reference" stands to *Grundgesetsze* in manner of presentation, except that in "On Sense and Reference," Frege disguises the mathematical implications. He is so successful in this that almost serendipitously he lays much of the foundations, both conceptually and technically, for a whole independent branch of philosophy, the philosophy of language. Indeed, his central argument is such a tour de force that it has stood the test of time as an analysis of oblique contexts in natural language, independently of its original purpose as an argument for the reality of *Sinn*. But we should not be misled by this achievement into mistaking what "On Sense and Reference" is about. It is an essay about identity statements. But to be about identity statements is to be about something much more fundamental for Frege - the nature of number. Throughout Frege keeps the enemy squarely in sight. Although they go unmentioned, it is the usual suspects - the mathematical formalists and their incoherent theory of number.

Frege begins his most famous essay by posing the question: is identity a relation?[72] There is a lure, Frege admits, to viewing identity as a relation between "names or signs of objects," a view by his own admission he was tempted in *Begriffsschrift*. The lure is to be found in the apparent avoidance of the problem, originally posed by the formalists, engendered by the opposite answer - identity as a relation holding of the object named - of being unable to distinguish true "$a = b$" from "$a = a$." So Frege asks us to suppose that identity is a relation between signs; under what conditions would it truly hold? Whatever the conditions are, they must involve what it is that the symbols have in common. For Frege this is the "same designated thing;" only if they have this in common could the relation be said to truly hold of the signs for that object. But nevertheless, even though $a = b$ and $a = a$ now appear to be different relations, we are really no further along with the problem. An object, Frege observes, may be symbolized in any number of ways; this is a fact of life. But if this is all that identity statements are about, there still will not be sufficient information to calculate a difference in

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the reader who is not frightened of the concept-script" (p. 120).

[72]In the text, Frege asks whether *equality* is a relation; in a footnote, he says that he "use[s] this word in the sense of identity." (p. 56).

cognitive value. To assess a difference in cognitive value, we must consider the *reason* for this multiplicity of symbols. If distinct signs correspond to distinct "modes of presentation" of a reference, only then is there good reason for distinct signs. A statement *that* an object has two names does not deliver the information *why* it has two names; but without such information an identity statement would be left bereft of cognitive value.

Frege is, of course, being a bit disingenuous when he poses this view as his prior view of *Begriffsschrift*. That analysis could not even be stated under the assumptions present at the time of "On Sense and Reference," since the notion of conceptual content is no longer defined. Content is now broken into two parts, sense and reference, so if there were to be an identity of content relation between expressions it would have to be either identity of sense - "— has the same sense as —" or identity of reference - "— has the same reference as —". The *Begriffsschrift* identity of content symbol ("≡") cannot, however, be identified with either of these. Identity of sense, where we gloss "has" as "express," is of course too strong, and Frege in fact nowhere defines such a symbol; thus our concern is to be with identity of reference. But identity of reference is not the same as identity of conceptual content. This is because only the latter relation is one that holds between expressions *qua* how their contribution to propositional content is determined by their associated modes of determination. The former notion, the one Frege criticizes in the opening paragraph of "On Sense and Reference," is simply a coreference relation; it holds of expressions detached from modes of presentation. The critical point is that if they are detached from modes of presentation, then they are also detached from cognitive values, and hence no distinctions can be drawn that turn of difference of cognitive value. Where there is no such connection of sense to reference there can be no difference in cognitive value. This is the conclusion Frege reaches in the following passage:[73]

> Nobody can be forbidden to use any arbitrarily producible event or object as a sign for something. In that case the sentence $a=b$ would no longer refer to the subject matter, but only to its mode of designation; we

[73]"On Sense and Reference," p. 57.

> would express no proper knowledge by its means. But in many cases this is just what we want to do. If the sign '*a*' is distinguished from the sign '*b*' only as object (here by means of its shape), not as sign (i.e. not by the manner in which it designates something), the cognitive value of *a*=*a* becomes essentially equal to that of *a*=*b*, provided that *a*=*b* is true. A difference can arise only if the difference between the signs corresponds to a difference in the mode of presentation of that which is designated.

"'*a*' is coreferential with '*a*'" and "'*a*' is coreferential with 'b'" both express thoughts; the former is *a priori*, the latter *a posteriori*. Hence, both "'*a*'" and "'*b*'" express senses. However, there is something missing from these senses: the mode of presentation part. We don't have modes of presentation of the things these sentences are about because these things are already present; sense doesn't determine reference because the reference, the sign itself, is already there. That is fine if our purpose is to say something about the linguistic devices by which we go about making statements about things. But what we require in the general case are not statements about statements about things, but statements about things themselves; with the former we are at a level too detached to express proper knowledge of objects. The point in the interior of a triangle at which lines from the vertices to the mid-point of the opposite side intersect may be designated as the intersection of any two of the three lines. "So," Frege says in concluding the opening paragraph, "we have different designations for the same point, and these names ('point of intersection of *a* and *b*,' 'point of intersection of *b* and *c*') likewise indicate the mode of presentation; and hence, the statement contains actual knowledge."[74]

[74]*Ibid*. We cannot make of Frege's choice here of a geometrical rather than arithmetic example what we did in *Begriffsschrift*. In the context of examples employed in other writings of this period, the rationale here appears to be more rhetorical than substantive; Frege would not have wanted to have been seen as begging the question, given that his goal was to independently motivate the key notion in an argument against he formalists notion of arithmetical equality. In the next paragraph he extends the geometric example to that of "the evening star" and "the morning star," but remains mum throughout on arithmetic.

The view that Frege is criticizing in commencing "On Sense and Reference" we can now see is not his own view of *Begriffsschrift*, but rather is a criticism of the view that he *criticizes* there. Consequently his remarks to this point in the essay are highly reminiscent of what he says in *Begriffsschrift* in isolating the role of "modes of determination" (*Bestimmungsweise*). However, much has changed in the ensuing years in Frege's comprehension of the significance of this notion, for what he has come to understand is that there is a way of capturing this notion in the theory of thought that supports a notion of objectual identity, a "relation between objects . . . in which each thing stands to itself but to no other thing." Rhetorically, Frege does not explicitly state that it is firmly establishing this point, and definitively refuting the formalists argument against equality understood as identity, that is the goal of the essay. Rather he begins with the second paragraph of the paper the process of leading us to this conclusion by laying out a theory of *signs*, those symbols that *express* a sense (*Sinn*), "wherein the mode of presentation is contained," and *stand for* or *designate* a reference (*Bedeutung*), the object so presented. The moves Frege then makes over subsequent paragraphs are well-known: First he establishes that senses can be references, and then, since references are objects, that senses are objects. His initial pass at showing this is with proper names; *prima facie,* their customary senses can be referred to, albeit indirectly, in reported speech with the locution 'the sense of the expression "A"'.[75] The second premiss is then established by deflecting the only alternative, that senses are ideas; ideas are undermined as references by their inherent subjectivity. Senses can't be ideas any more than numbers can be; if they were, then they would not have the sorts of properties that we expect references to have.[76] Frege now takes a second pass at the argument, this time with much more force and detail, turning to natural

[75]Russell's "Grey's Elegy" argument of "On Denoting" can be understood as attacking Frege at this point; cf. Simon Blackburn and Alan Code, "The Power of Russell's Criticism of Frege: 'On Denoting' pp. 48-50." *Analysis*, XXXVIII, 2 (1978): 65-77. On Blackburn and Code's interpretation, the problem Russell directs towards Frege (and his own earlier views) is that while the description in question denotes a sense, there is no way to give a meaning that *determines* that sense as the denotation; to put it in Fregean terms, there is no indirect sense to determine customary sense as indirect reference.

[76]Cf. *Grundlagen*, §§25-27

language. For declarative sentences, Frege argues, their sense is a thought, their reference, a truth-value. This much is established by a substitution argument; the latter, but not the former, remains constant under substitution of coreferential parts of a sentence. But there are telling exceptions, contexts in which truth-value does not remain constant; these exceptions can be isolated grammatically as sentences that themselves contain subordinate clauses (oblique contexts). For the remainder of the essay up to the closing paragraph, Frege explores these exceptions in great depth and with considerable linguistic sophistication, establishing that they fall under the following principle:[77]

> In such cases it is not permissible to replace one expression in the subordinate clause by another having the same customary reference, but only by one having the same indirect reference, i.e. the same customary sense.

For Frege, the central tenet is that anything that is *Bedeutung* is objective; in oblique contexts, senses are *Bedeutung*, given the substitution argument; hence, senses are objects.

Having put all the pieces into place, Frege is now ready to deliver the punch line in the final paragraph of the paper:[78]

> When we found ‘*a*=*a*’ and ‘*a*=*b*’ to have different cognitive values, the explanation is that for the purpose of knowledge, the sense of the sentence, viz., the thought expressed by it, is no less relevant than its reference, i.e. its truth value. If now *a*=*b*, then indeed the reference of ‘*b*’ is the same as that of ‘*a*,’ and hence the truth value of ‘*a*=*b*’ is the same as ‘*a*=*a*.’ In spite of this, the sense of ‘*b*’ may differ from that of ‘*a*’, and thereby the thought expressed in ‘*a*=*b*’ differs from that of ‘*a*=*a*.’ In that case the two sentences do not have the same cognitive value. If we understand by ‘judgment’

[77]“On Sense and Reference,” p. 67.

[78]“On Sense and Reference,” p. 78.

> the advance from the thought to its truth value, as in the above paper, we can also say that the judgments are different.

At the end of his most justifiably famous paper, Frege can give this argument, with all its implications, with the firm conviction that he has argued as compellingly as possible to place the central notion on which it turns - *Sinn* - on firm foundational footings. In his letter to Peano that we cited above, Frege complains that:[79]

> As far as the equals sign is concerned, your remark that different authors have different opinions about its meaning leads to considerations that very many mathematical propositions present themselves as equations and that others at least contain equations, and if we place this against your remark, we get the result that mathematicians agree indeed on the external form of their propositions but not on the thoughts they attach to them, and these are surely what is essential. What one mathematician proves is not the same as what another understand by the same sign. This is surely an intolerable situation which must be ended as quickly as possible.

There is only one possible meaning of equality, according to Frege, that can restore the peace: identity. "On Sense and Reference" is the final piece of the puzzle, presented in a precise and decisive, yet non-technical, way, for establishing the identity theory of *Grundgesetsze*. While that theory is distinct from that of *Begriffsschrift*, at its heart is an insight with its origins in the earlier monograph; what Frege finds in his mature work of the late 19th and early 20th centuries is a way of expressing this insight semantically that allows him to dismiss outright what he viewed, in contrast to his own views, as the incoherent and chaotic view of number held by many of the most influential mathematicians of his day. With the machinery of sense and reference, no doubt Frege's most important bequest to modern thought, the account of identity statements

[79] Letter to Peano, p. 126.

he developed to accomplish this is so brilliant and full of both philosophical and linguistic insight, that it has outlasted the demise of both his view of number and many of the assumptions that animated his philosophical outlook. How we are to evaluate Frege's account once removed from this context, (especially with respect to contemporary skepticism about sense), is a matter we leave for the sequel.

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