Notes on Frege on Rules of Inference

R. May

1. *There is only one rule of inference, modus ponens.* This is true both in the presentations of *Begriffsschrift* and *Grundgesetze*. (But cf. note regarding the latter.) There are other ways of making transitions between propositions in proofs, but these are never labeled by Frege "rules of inference." These pertain to scope of quantification, parsing of formulas (bracketing), introduction of definitions, conventions for the use and replacement of the various letters(variables), and certain structural reorganizations, (e.g. amalgamation of horizontals, and of identical subcomponents); cf. the list in *Gg* §48.

2. Modus ponens holds a privileged place not just because it can lead to new judgements, but in the manner in which it does so, by transforming what was previously only a judgeable content, (contained as the consequent of a conditional), into a judgement. This is important, because it is the key to how proof can give rise to new knowledge, (which may be analytic); by modus ponens steps content that may only be implicitly contained in the basic laws can be made explicit. A transition under equivalence, while it may map judgements onto judgements, cannot lead to a judgement with a new content. This is particularly clear in *Bg*, where equivalence is identity of content.

3. Frege's remark in *Begriffsschrift* §6 - "I use just this one - at least in all cases where a new judgement is derived from more than one single judgement" - is to be understood in the way of advertisement by calling attention to a main economy and hence advantage of his system. What Frege is saying is that given the multitude of Aristotelean modes of inference, all the inferences that are covered by these are covered by one rule of inference - modus ponens - at least insofar as these involve proving a new judgement from two or more other judgements. The last proviso is just that if there really is a proof from one judgement, inherently modus ponens wouldn't be applicable. So Frege's remark does not preclude that there are other rules of inference.

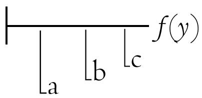
4. However, in other places Frege does clearly imply that he thinks there is only one rule of inference. For example, in "Boole's Logical Calculus and the Concept-script," (1880/1) in which he goes to some length to describe the characteristics of his system, he lists the "rules set out in words"; second in the list is "The rule of inference," (in the singular). (While Frege clearly maintains the uniqueness of modus ponens at the time of *Begriffsschrift*, he appears to increase the extension of rules of inference in *Grundgesetze*; but see note 9.)

5. The format of proofs for Frege is significant, (just as is the format of propositions). Proofs as they are laid out in *Bg* are divided into two columns. On the left are tables, on the right an ordered list of propositions (judgements). The role of the tables is to specify the contents of the propositions so as to validate a modus ponens step in the proof; "the little table . . . serves to make [the] proposition . . . more easily recognizable in the more complex from in which it appears here." (*Bg*, 142) The successive steps giving the list of propositions are all (and only) by modus ponens. (But see note 9 for differences in *Gg*.)

6. In *Bg*, the tables are governed by general laws of quantification. Given Frege's convention that italic letters are implicitly wide-scope (universal) quantifiers, the "substitutions" specified in the tables are

either (i) uniform re-letterings or (ii) universal instantiation, i.e. replacement of an italic letter by an instance falling under the universal generalization. Because the variables in Frege's system range over everything, this gives substitution an appropriate generality, but not without its costs, given that Frege's logic is full impredicative second-order logic.

7. A given table may encapsulate a complex of inferential steps; all the steps that are needed to obtain a proposition that allows for the next modus ponens step in the proof. Some of these steps may be implicit. In *Begriffsschrift*, Frege derives proposition (81) from (80) and (74), with the substitutions specified for the latter by "$y \mid \mathfrak{a}$". (74) has the form:



and there are at least four steps needed to obtain the proper form. (1) Replace the italic with the gothic, and place a wide-scope divot - "placed immediately after the judgement stroke." (*Bg*, 132). (2) Turn the wider scope conditional into a conjunction, as described in *Bg*, 133 - i.e. $a \supset (b \supset c)$ turns into $(a \,\&\, b) \supset c$. (3) Move the scope of the divot inside the conjunction, by the procedure described in *Bg*, 132. (4) Convert the conjunction back into a conditional; again cf. *Bg*, 133. These "transitions" now result in a form that can be the (suppressed) minor premise in the modus ponens step to (81), because the scope of the quantifier has been given the appropriately narrower scope. None of the transitions, howver, gives rise to new propositions on the right. (Note the "at least" - this is because I have not counted the steps needed to obtain the equivalence relied upon in (2) and (4).)

8. In *Begriffsschrift,* Frege remarks that other rules of inference may be added later, so that "transitions from several judgements to a new one, which are possible by this single mode of inference in only an indirect way, be converted into direct ways for the sake of abbreviation . . . In this way, then, further modes of inference would arise." (*Bg*, 107) Frege contemplates doing this early on; he remarks in "Boole's Logical Calculus and the Concept-script" that "This is achieved by converting what was expressed in a formula into a rule of inference." The case he has in mind is proposition (52) of *Begriffsschrift*:

(52) $c \equiv d \supset (f(c) \supset f(d))$,

turned into a rule for substitution under identity. So while given $c \equiv d$ and $f(c)$, $f(d)$ follows from (52) via modus ponens, we can now go directly from the first two premisses to $f(d)$. This allows for the abbreviation of proofs. In *Grundgesetze*, Frege does not proceed in this manner, giving an analogue of (52) as Basic Law III. But he does employ this strategy in *Gg* in two cases, contraposition and interchangability of subcomponents, which in *Bg* are "kernal" propositions (i.e. basic laws). (Contraposition is proposition (28); interchangability of subcomponents is proposition (8).) Nevertheless, Frege refrains from labeling either of these rules of *inference*, even though in *Gg* contraposition its own transition sign.

9. In *Grundgesetze,* Frege specifies three rules as rules of inference, the first of which is (a generalized by incorporating interchangeability of subcomponents form of) modus ponens. The second rule is (a generalized form of) the transitivity of the conditional, and the third specifies (a generalized form of) ex-

cluded middle, the inference from $\neg a \supset b$ and $a \supset b$ to b. (Cf. *Gg*, §§14 - 16; they are distinguished from other rules involved in proof in §48, "Summary of the Rules.") Each of these three rules of inference in *Gg* gives rise to new propositions on the right side of proofs; each is associated with it own typographically distinct transition sign. Both of the latter two rules, however, are abbreviatory for longer inferences. Frege is particularly explicit about this for the excluded middle in *Gg*, (a result of transitivity, contraposition and amalgamation), and in *Bg* shows how transitivity follows; cf. the proof of proposition (5) from kernal propositions (1) and (2), (the latter is a theorem of *Gg*). Unlike contraposition, however, these rules are not re-phrasings of basic laws as rules; rather they specify shortened forms of proofs that could otherwise be carried out by modus ponens as the only rule of inference, (along with other rules that are not rules of inference).

10. The logic of *Begriffsschrift* compares with that of *Grundgezetse* in that the former has more basic laws and fewer rules, the latter has fewer laws and more rules. This allows in *Gg* for more economical proofs, with, Frege thought, a resulting increase in clarity and perspicuity, but with a loss of the more pristine organizations of proofs of *Bg*, in which each transition from proposition to proposition on the right is via modus ponens. C. Thiel says: "Where sentences *in* the *Begriffsschrift* are to be proved, the process will be essentially shortened by the introduction of other forms. In investigations *on* the *Begriffsschrift* it will be simpler to have a single form of inference." What Thiel presumably means here is that if we wish to know how the logical system *works*, then in some fundamental sense, in both *Bg* and *Gg* there is only one rule of inference - modus ponens. But if we are using the logic, then we may codify certain shortcuts in order to make the proofs as compact as possible, while still ensuring that no steps have been missed, that there are no gaps.

11. So we are still left with the question of why Frege maintained that there was only one rule of inference. If the remarks in note 2 are along the right lines, the justification will be epistemological, although one might inquire whether the answer would be the same in the contexts of *Bg* and *Gg*. There is some reason to think that they would differ, at least in form if not substance. This is because by the time of *Grundgesetze*, logic does not have to carry the sort of epistemic load it did in *Begriffsschrift* and *Grundlagen*, given the rise of *Sinn* as the central epistemological notion for Frege. Modus ponens, under the interpretation of the notation in *Bg* literally mapped from judgeable content to judgement. But in *Gg* the relevant parts of the notation are stripped of these meanings; they are now merely "the vertical" and "the horizontal." So if modus ponens is to lead us, by dint of logic, from unjudged to judged, it must do so in the circumstances of how judgement is to be understood in the context of sense and reference.

13 May2002